Poincaré Dodecahedral Space

MathsJam 2012

Nicholas Jackson

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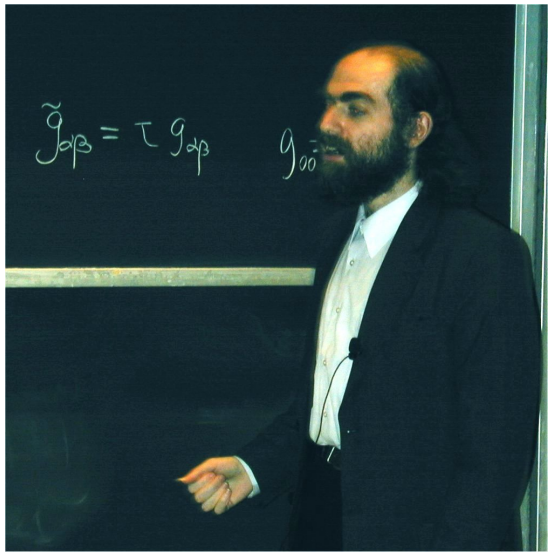
Henri Poincaré

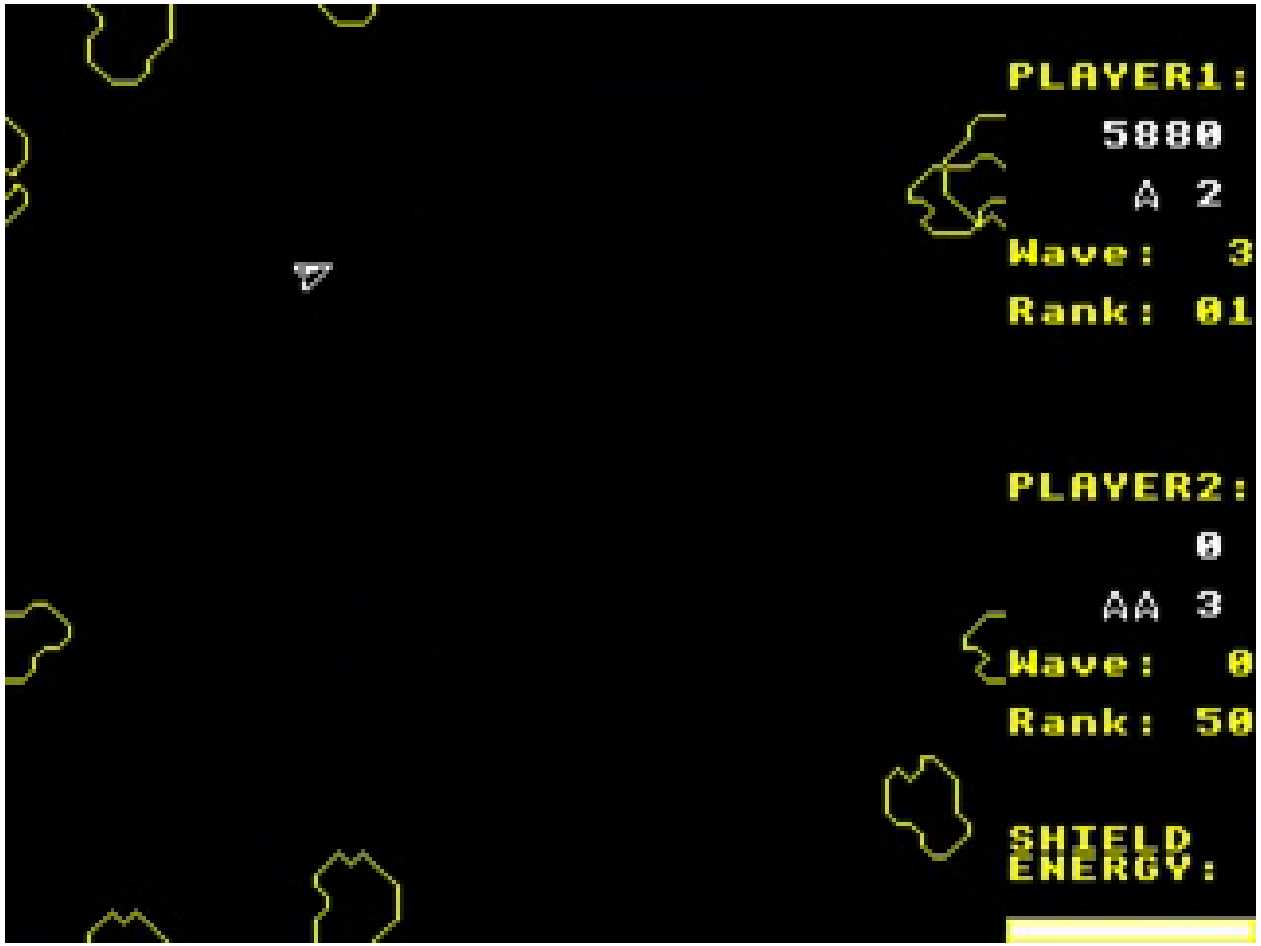




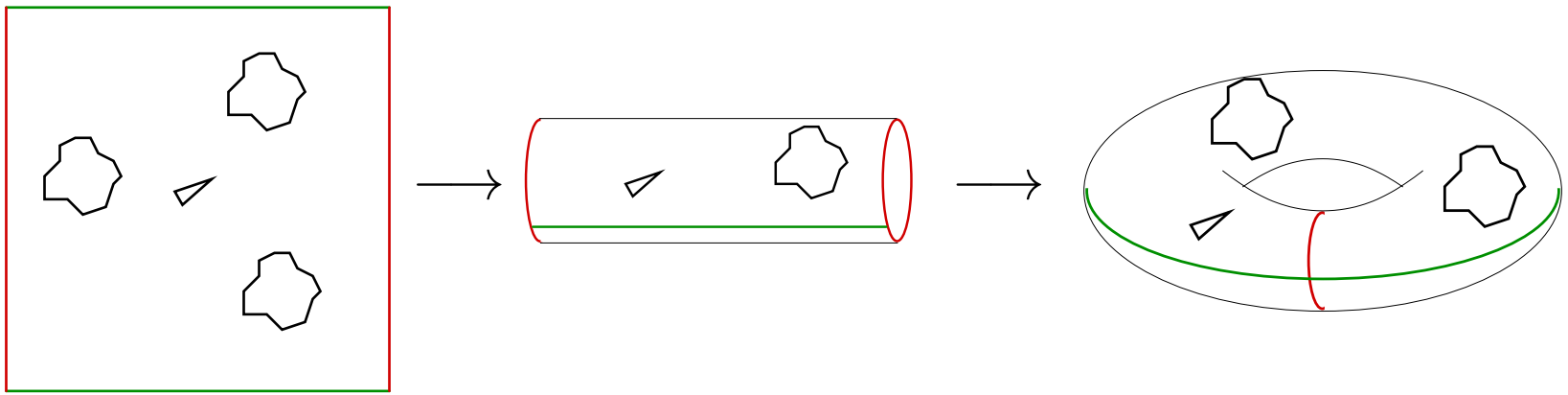
Jules Henri Poincaré (1854–1912)

- Mathematician, theoretical physicist, mining engineer and philosopher of science: "The Last Universalist".
- Work on electromagnetism, non-Euclidean geometry, number theory, dynamics (the three-body problem), relativity, ...
- Pioneered algebraic topology (analysis situs) – use of algebraic (group theoretic) methods to solve topological problems.
- Cousin of Raymond Poincaré, President of France 1913–1920.

Asteroids


PLAYER1:
5880
A 2
Wave: 3
Rank: 01
PLAYER2:
0
AA 3
Wave: 0
Rank: 50
SHIELD
ENERGY:

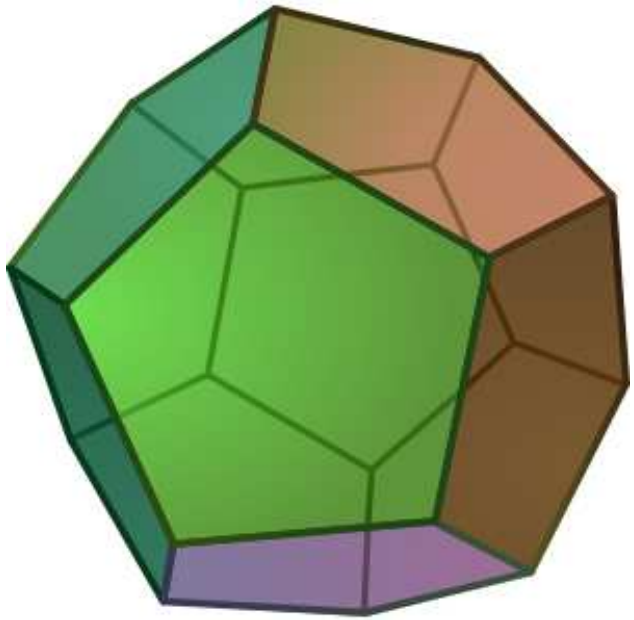
Glue (identify) opposite sides to get a torus.



From the point of view of the spaceship, this looks like ordinary 2–dimensional space, except that it folds back on itself.

Poincaré Dodecahedral Space

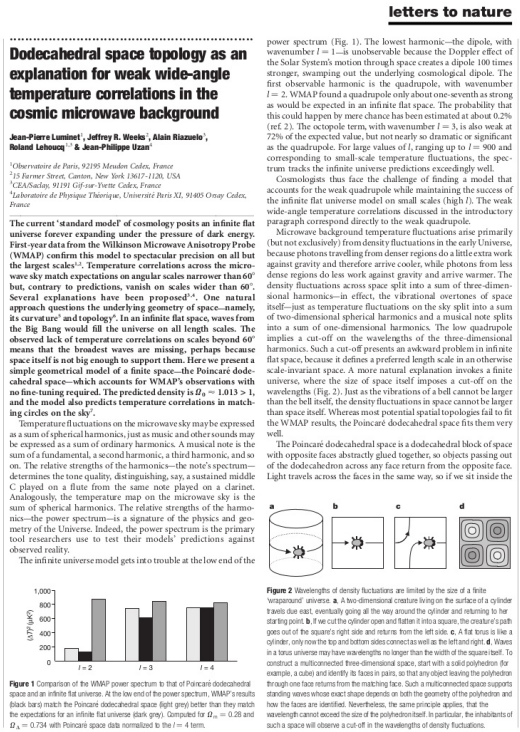
Glue opposite faces with a $\frac{1}{10}$ $(36^\circ = \frac{\pi}{5})$ twist.



- Three faces meet at each edge.
- Four edges meet at each vertex.
- Five "different" vertices.

From the spaceship's point of view this looks like ordinary 3–dimensional space, except that it folds back on itself and is (positively) curved.

Our universe might be a Poincaré dodecahedral space.



letters to nature

Dodecahedral space topology as an explanation for weak wide-angle temperature correlations in the cosmic microwave background

Jean-Pierre Luminet[1], Jeffrey R. Weeks[2], Alain Riazuelo[3], Roland Lehoucq[1,4] & Jean-Philippe Uzan[3]

[1]*Observatoire de Paris, 92195 Meudon Cedex, France*
[2]*15 Farmer Street, Canton, New York 13617-1120, USA*
[3]*CEA/Saclay, 91191 Gif-sur-Yvette Cedex, France*
[4]*Laboratoire de Physique Théorique, Université Paris XI, 91405 Orsay Cedex, France*

The current 'standard model' of cosmology posits an infinite flat universe forever expanding under the pressure of dark energy. First-year data from the Wilkinson Microwave Anisotropy Probe (WMAP) confirm this model to spectacular precision on all but the largest scales[1,2]. Temperature correlations across the microwave sky match expectations on angular scales narrower than 60° but, contrary to predictions, vanish on scales wider than 60°. Several explanations have been proposed[3–6]. One natural approach questions the underlying geometry of space—namely, its curvature[7] and topology[8]. In an infinite flat space, waves from the Big Bang would fill the universe on all length scales. The observed lack of temperature correlations on scales beyond 60° means that the broadest waves are missing, perhaps because space itself is not big enough to support them. Here we present a simple geometrical model of a finite space—the Poincaré dodecahedral space—which accounts for WMAP's observations with no fine-tuning required. The predicted density is $\Omega_0 \approx 1.013 > 1$, and the model also predicts temperature correlations in matching circles on the sky[9].

Temperature fluctuations on the microwave sky may be expressed as a sum of spherical harmonics, just as music and other sounds may be expressed as a sum of ordinary harmonics. A musical note is the sum of a fundamental, a second harmonic, a third harmonic, and so on. The relative strengths of the harmonics—the note's spectrum—determines the tone quality, distinguishing, say, a sustained middle C played on a flute from the same note played on a clarinet. Analogously, the temperature map on the microwave sky is the sum of spherical harmonics. The relative strengths of the harmonics—the power spectrum—is a signature of the physics and geometry of the Universe. Indeed, the power spectrum is the primary tool researchers use to test their models' predictions against observed reality.

The infinite universe model gets into trouble at the low end of the power spectrum (Fig. 1). The lowest harmonic—the dipole, with wavenumber $l = 1$—is unobservable because the Doppler effect of the Solar System's motion through space creates a dipole 100 times stronger, swamping out the underlying cosmological dipole. The first observable harmonic is the quadrupole, with wavenumber $l = 2$. WMAP found a quadrupole only about one-seventh as strong as would be expected in an infinite flat space. The probability that this could happen by mere chance has been estimated at about 0.2% (ref. 2). The octopole term, with wavenumber $l = 3$, is also weak at 72% of the expected value, but not nearly so dramatic or significant as the quadrupole. For large values of l, ranging up to $l = 900$ and corresponding to small-scale temperature fluctuations, the spectrum tracks the infinite universe predictions exceedingly well.

Cosmologists thus face the challenge of finding a model that accounts for the weak quadrupole while maintaining the success of the infinite flat universe model on small scales (high l). The weak wide-angle temperature correlations discussed in the introductory paragraph correspond directly to the weak quadrupole.

Microwave background temperature fluctuations arise primarily (but not exclusively) from density fluctuations in the early Universe, because photons travelling from denser regions do a little extra work against gravity and therefore arrive cooler, while photons from less dense regions do less work against gravity and arrive warmer. The density fluctuations across space split into a sum of three-dimensional harmonics—in effect, the vibrational overtones of space itself—just as temperature fluctuations on the sky split into a sum of two-dimensional spherical harmonics and a musical note splits into a sum of one-dimensional harmonics. The low quadrupole implies a cut-off on the wavelengths of the three-dimensional harmonics. Such a cut-off presents an awkward problem in infinite flat space, because it defines a preferred length scale in an otherwise scale-invariant space. A more natural explanation invokes a finite universe, where the size of space itself imposes a cut-off on the wavelengths (Fig. 2). Just as the vibrations of a bell cannot be larger than the bell itself, the density fluctuations in space cannot be larger than space itself. Whereas most potential spatial topologies fail to fit the WMAP results, the Poincaré dodecahedral space fits them very well.

The Poincaré dodecahedral space is a dodecahedral block of space with opposite faces abstractly glued together, so objects passing out of the dodecahedron across any face return from the opposite face. Light travels across the faces in the same way, so if we sit inside the

Figure 2 Wavelengths of density fluctuations are limited by the size of a finite 'wraparound' universe. **a**, A two-dimensional creature living on the surface of a cylinder travels due east, eventually going all the way around the cylinder and returning to her starting point. **b**, If we cut the cylinder open and flatten it into a square, the creature's path goes out of the square's right side and returns from the left side. **c**, A flat torus is like a cylinder, only now the top and bottom sides connect as well as the left and right. **d**, Waves in a torus universe may have wavelengths no longer than the width of the square itself. To construct a multiconnected three-dimensional space, start with a solid polyhedron (for example, a cube) and identify its faces in pairs, so that any object leaving the polyhedron through one face returns from the matching face. Such a multiconnected space supports standing waves whose exact shape depends on both the geometry of the polyhedron and how the faces are identified. Nevertheless, the same principle applies, that the wavelength cannot exceed the size of the polyhedron itself. In particular, the inhabitants of such a space will observe a cut-off in the wavelengths of density fluctuations.

Figure 1 Comparison of the WMAP power spectrum to that of Poincaré dodecahedral space and an infinite flat universe. At the low end of the power spectrum, WMAP's results (black bars) match the Poincaré dodecahedral space (light grey) better than they match the expectations for an infinite flat universe (dark grey). Computed for $\Omega_{tot} = 0.36$ and $\Omega_\Lambda = 0.734$ with Poincaré space data normalized to the $l = 4$ term.

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