Coin tossing sequences

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- Toss a coin repeatedly until we get a particular sequence.
- e.g. HTT

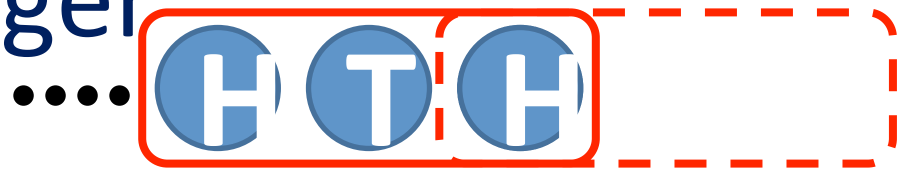
9 tosses

- How many tosses on average?
- Is it the same for all sequences?

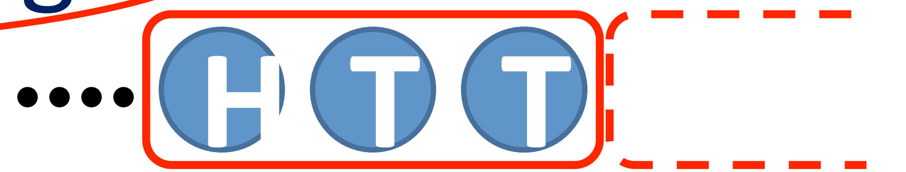
How many tosses on average to get HTT or HTH?

Options:

A. HTT takes longer



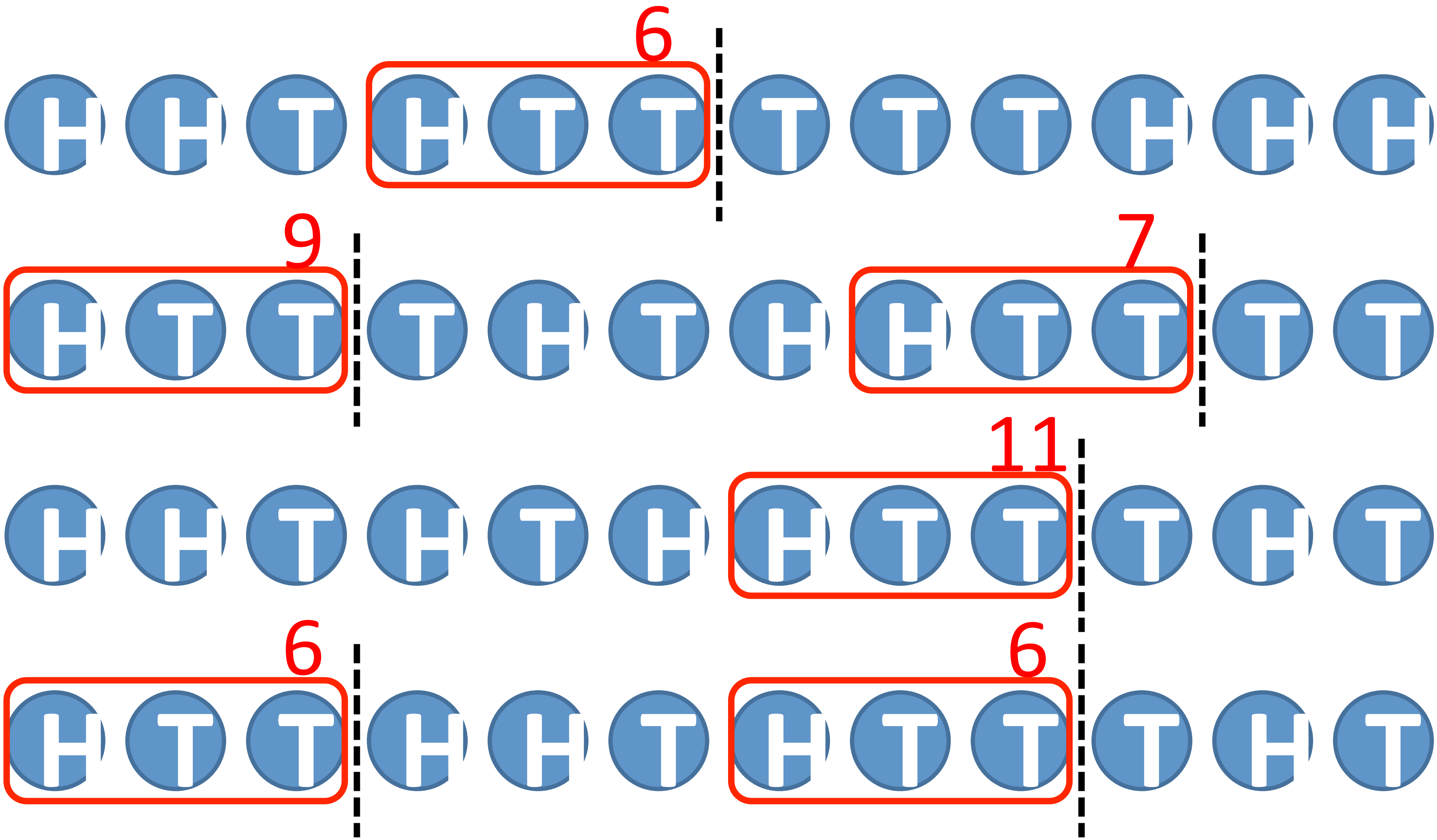
B. HTH takes longer



C. Both the same

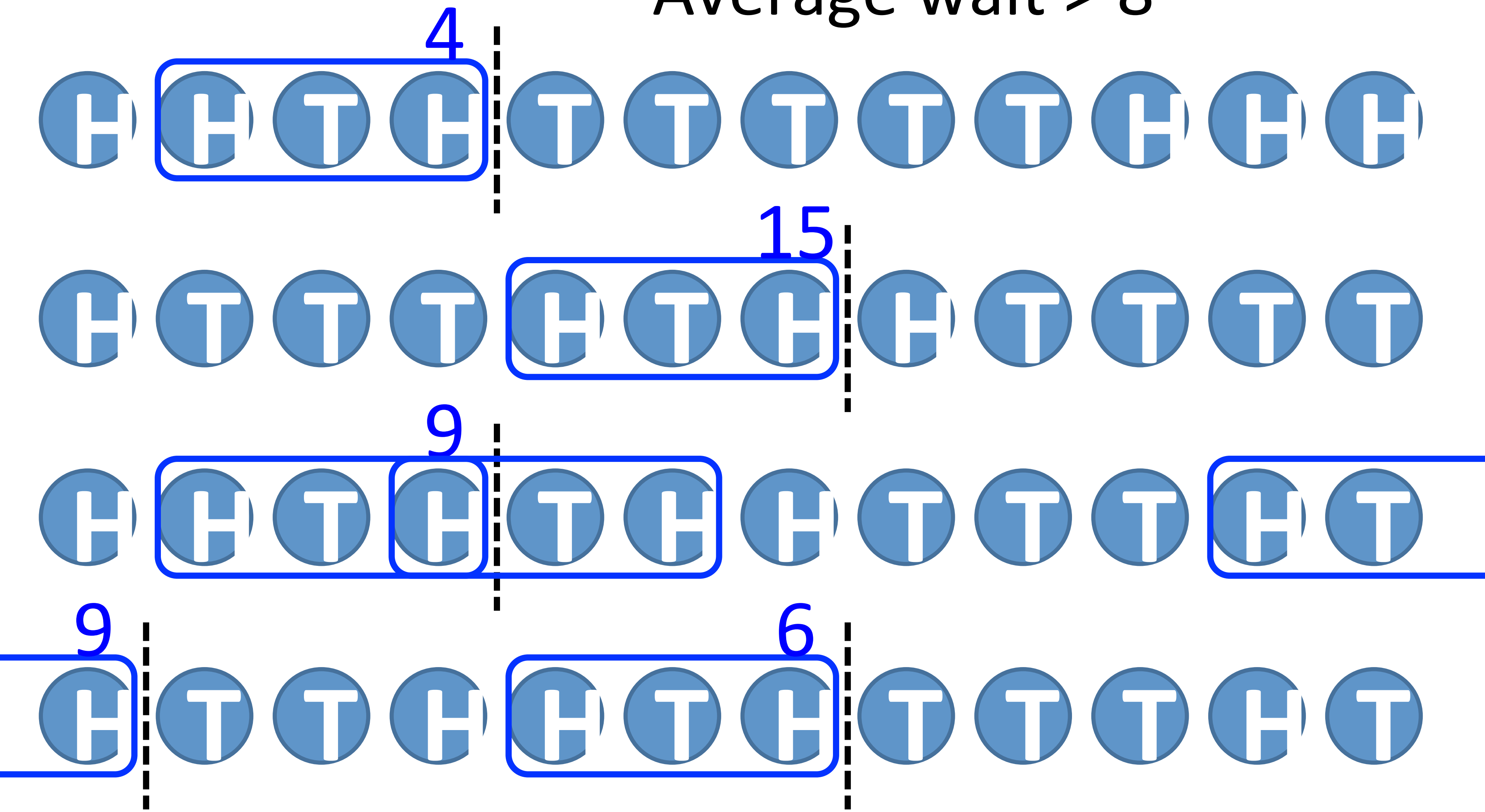
Target: HTT

Probability = 1/8
Average wait = 8



Target: HTH

Probability = 1/8
Overlaps don’t get counted
Average wait > 8



Calculating the average wait

Target: HTH

	T	H	H	T	T	H	T	H	
Stake	1	1	1	1	1	1	1	1	**8**
Payout	0	0	0	0	0	8	0	2	**10**

- Bet £1 on each coin being start of chosen sequence
- In a fair game, payout for matching all 3 is £8
- Also payout £4 for matching 2, £2 for matching 1
- For HTH, payout = £10
- For fair game, average stake = payout

∴ Average number of tosses = 10 for HTH

Calculating the average wait

Target: HTH

	T	H	T	T	H	T	H	
Stake (£)	1	1	1	1	1	1	1	**7**
Payout (£)	0	0	0	0	8	0	2	**10**

- Bet £1 on each coin being start of chosen sequence
- In a fair game, payout for matching all 3 is £8
- Also payout £4 for matching 2, £2 for matching 1
- For HTH, payout = £10
- For fair game, average stake = payout

∴ Average number of tosses = 10 for HTH

Average wait

Sequence	Average wait
H	2
T	2

Sequence	Average wait
HH	6
HT	4
TH	4
TT	6

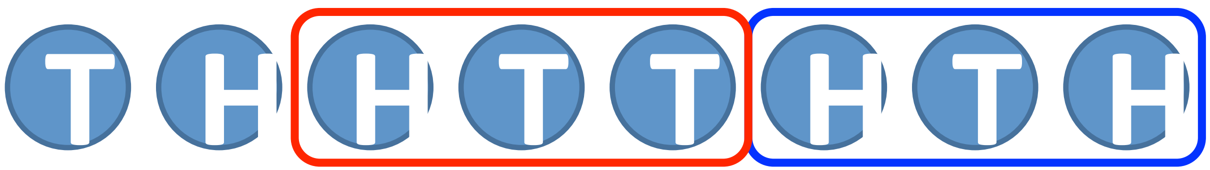
Sequence	Average wait
HHH	14
HHT	8
HTH	10
HTT	8
THH	8
THT	10
TTH	8
TTT	14

Sequence length *n*

- Minimum wait = 2^n
- Maximum wait = $2^{n+1}-2$
- Longer sequence always has longer wait

Two sequences

- Which is more likely to occur first?
- e.g. HTT, HTH



- Penney's game

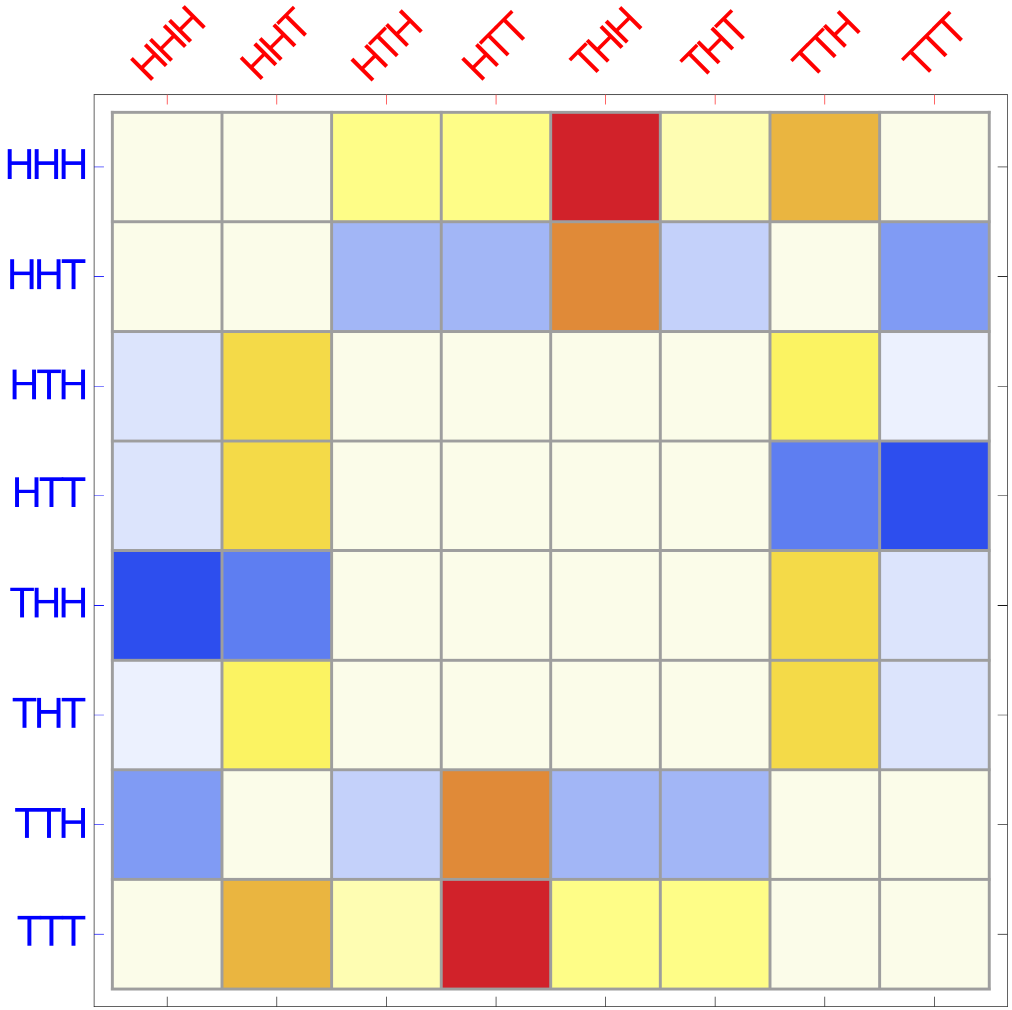
Walter Penney, Journal of Recreational Mathematics, October 1969, p.241

Probability of red sequence preceding blue

	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
HHH		1/2	3/5	3/5	7/8	7/12	7/10	1/2
HHT	1/2		1/3	1/3	3/4	3/8	1/2	3/10
HTH	2/5	2/3		1/2	1/2	1/2	5/8	5/12
HTT	2/5	2/3	1/2		1/2	1/2	1/4	1/8
THH	1/8	1/4	1/2	1/2		1/2	2/3	2/5
THT	5/12	5/8	1/2	1/2	1/2		2/3	2/5
TTH	3/10	1/2	3/8	3/4	1/3	1/3		1/2
TTT	1/2	7/10	7/12	7/8	3/5	3/5	1/2	

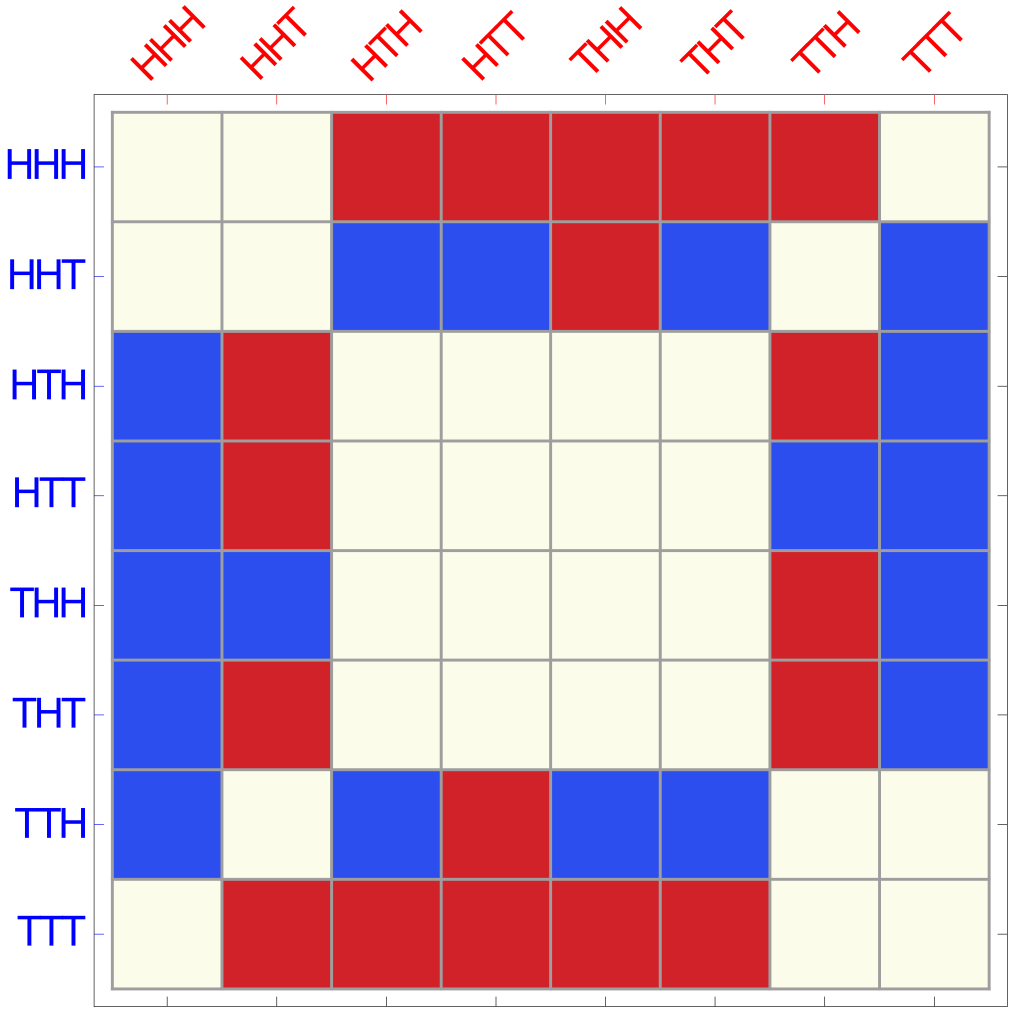
$n=3$

Probability of red sequence preceding blue



n=3

Which sequence is more likely to occur first

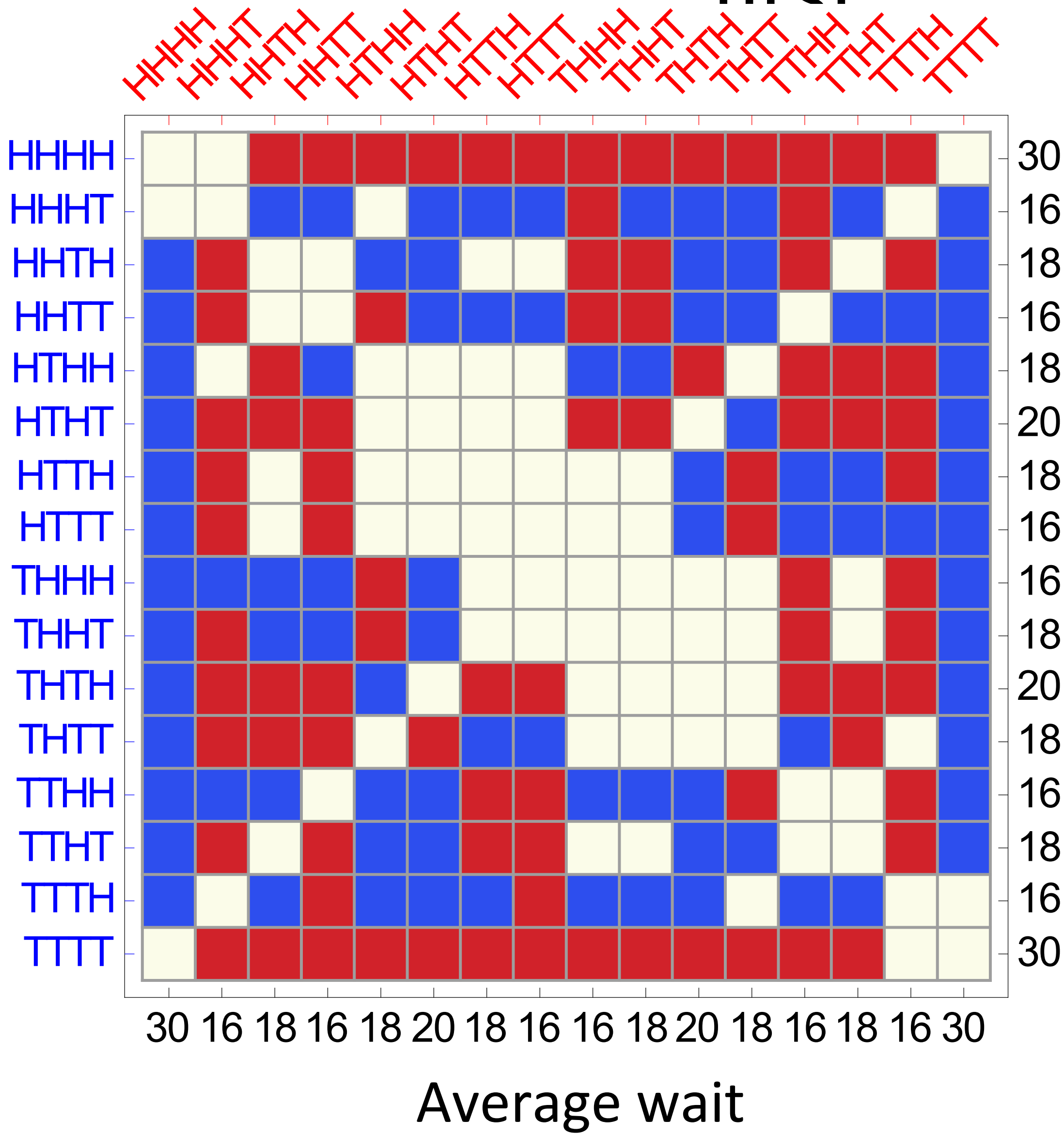


Average wait

n=3

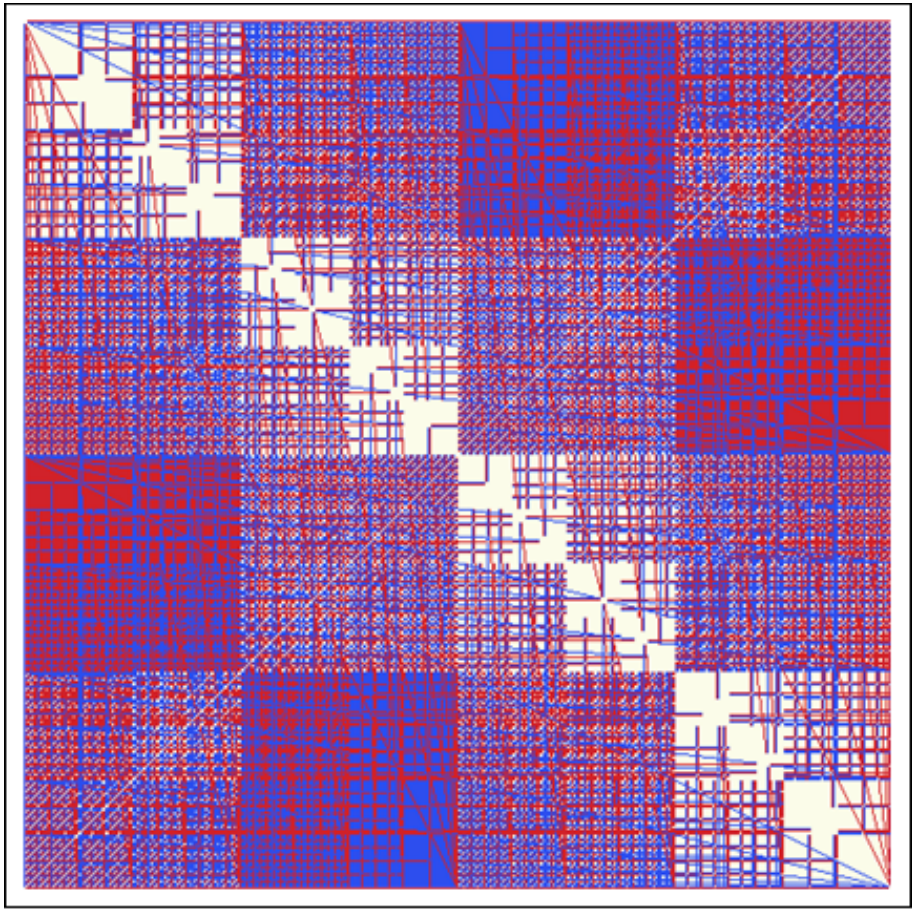
Shorter wait always at least 50% probability of preceding longer one

Which sequence is more likely to occur first



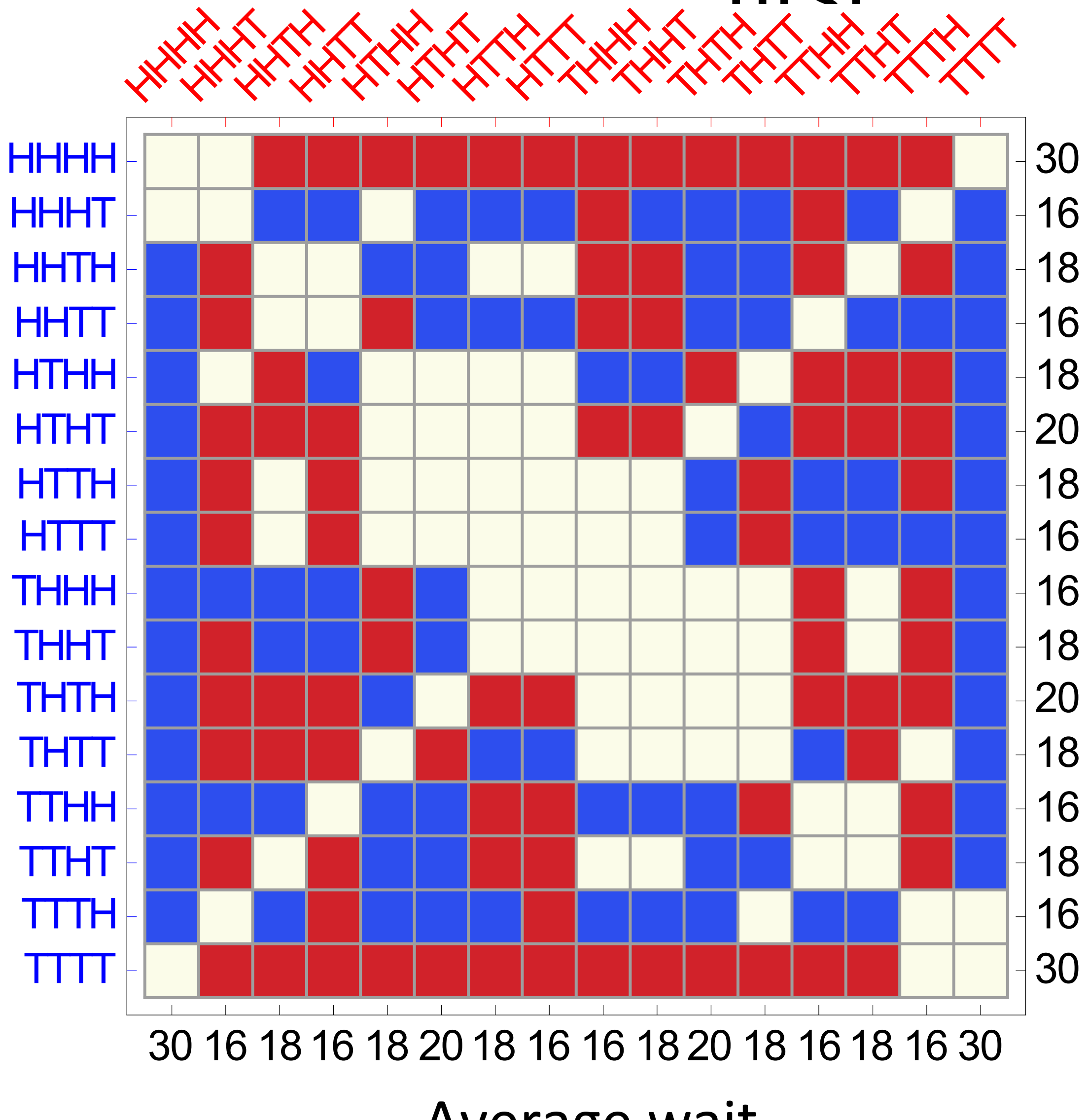
n=4

Which sequence is more likely to occur first



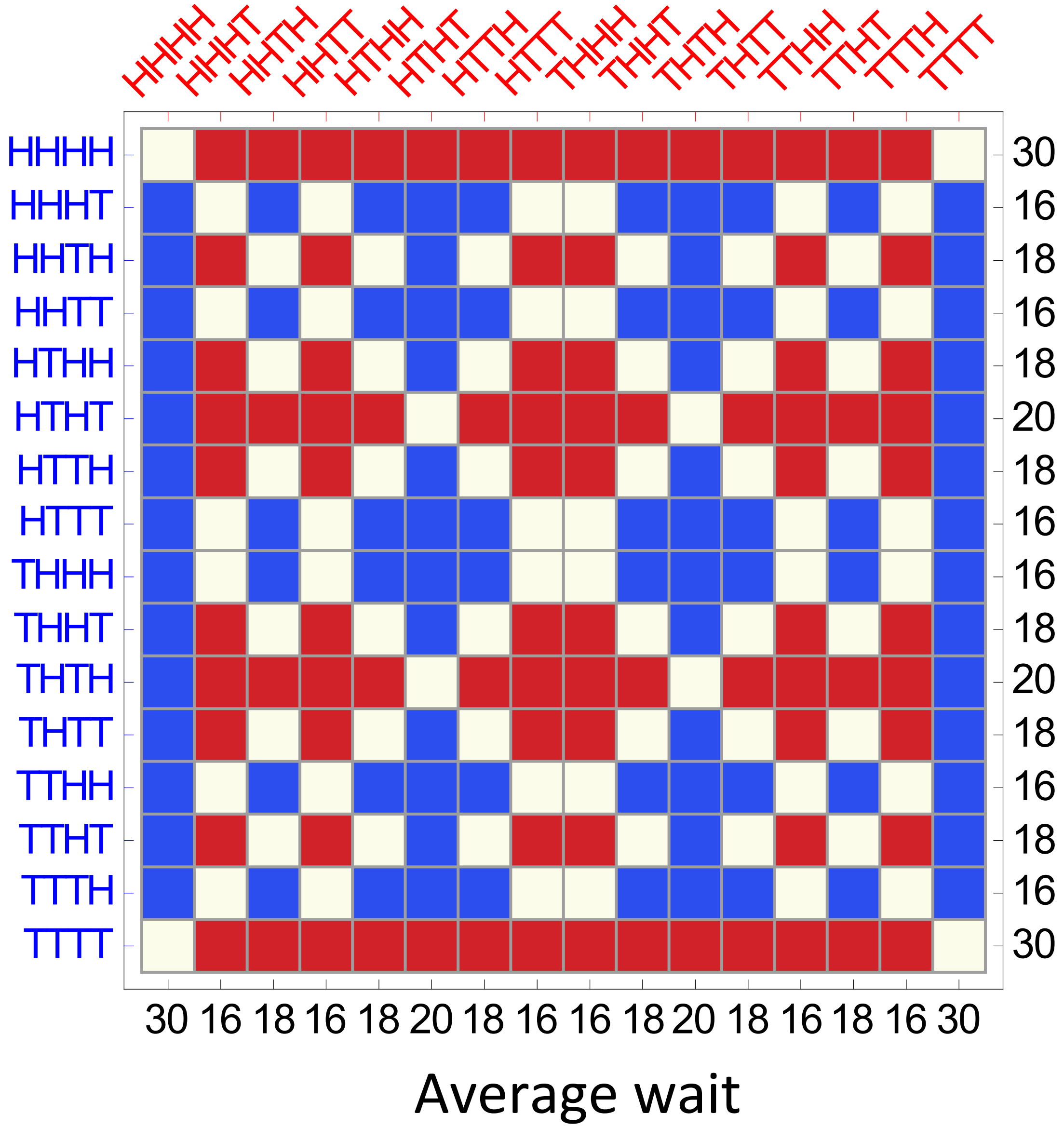
n=9

Which sequence is more likely to occur first



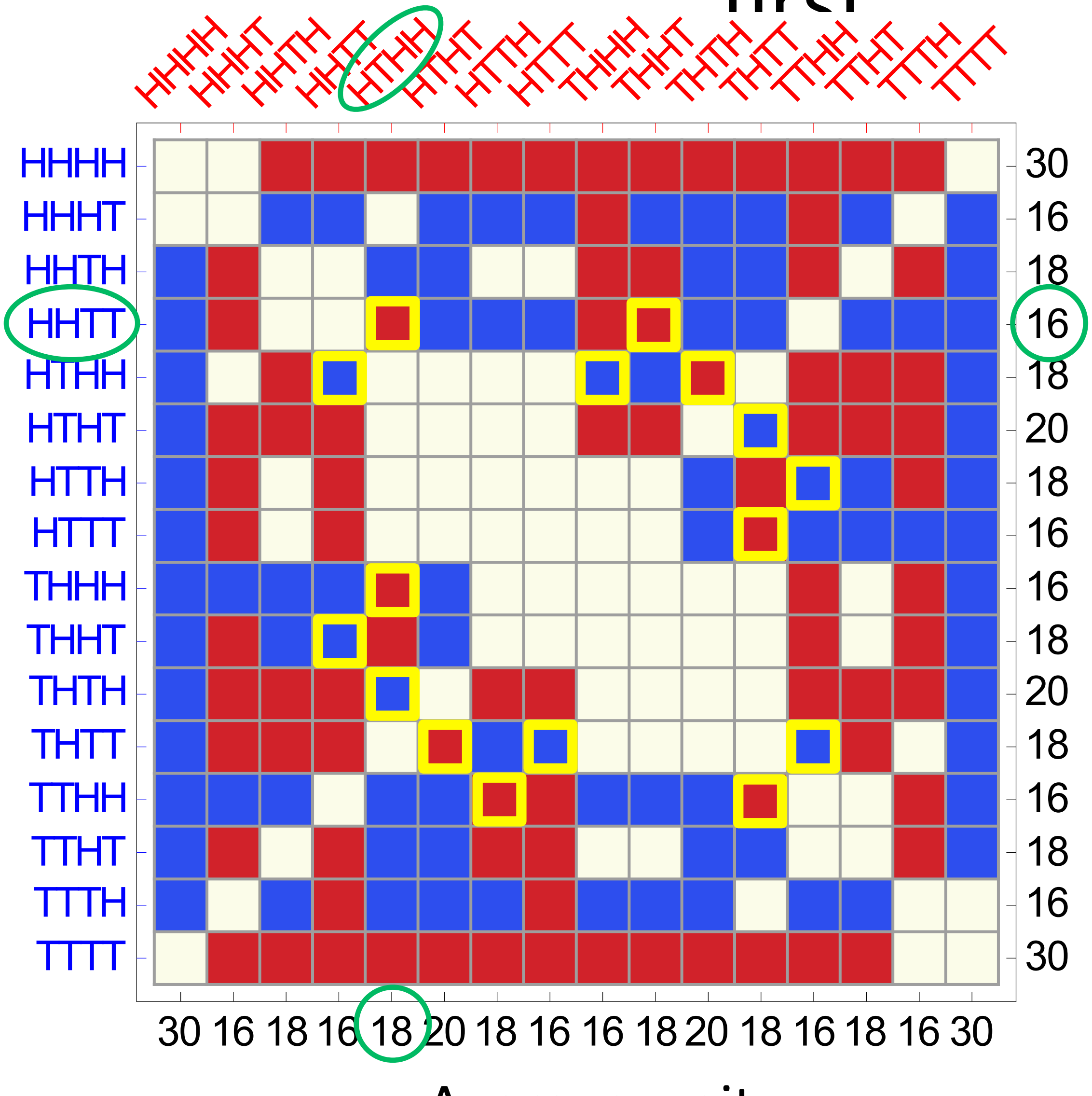
n=4

Which sequence has the shorter waiting time



n=4

Which sequence is more likely to occur first



n=4

Longer average wait is more likely to precede shorter one!

Shorter wait can precede longer wait

Ignoring transposition of H,T, distinct cases are:

- HTHH (18) beats HHTT (16) with $p=4/7 \approx 0.57$
- HTHH (18) beats THHH (16) with $p=7/12 \approx 0.58$
- THHT (18) beats HHTT (16) with $p=7/12 \approx 0.58$
- THTH (20) beats HTHH (18) with $p=9/14 \approx 0.64$

THTH has an almost 2/3 probability of preceding HTHH, yet takes longer to occur on average!

References

- Martin Gardner, Time Travel and other Mathematical Bewilderments, 60-66
- https://plus.maths.org/content/os/issue55/features/nishiyama/index
- Same article, but also including proof based on average wait time: http://www.i-repository.net/il/user_contents/02/G0000031Repository/repository/keidaironshu_063_004_269-276.pdf
- Theorems X and Y in this article can be seen to be true based on the gambling approach shown in the following
- Martingale approach: http://projecteuclid.org/download/pdf_1/euclid.aop/1176994578
- Which sequence will occur first? http://www.agenarisk.com/resources/probability_puzzles/equal_sequences.shtml
- Penney Ante: Counterintuitive Probabilities in Coin Tossing bact.mathcircles.org/files/Summer2010/PenneyAnte.pdf
- Penney's game https://en.wikipedia.org/wiki/Penney%27s_game
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- How to win at coin flipping http://blog.wolfram.com/2010/11/30/how-to-win-at-coin-flipping/