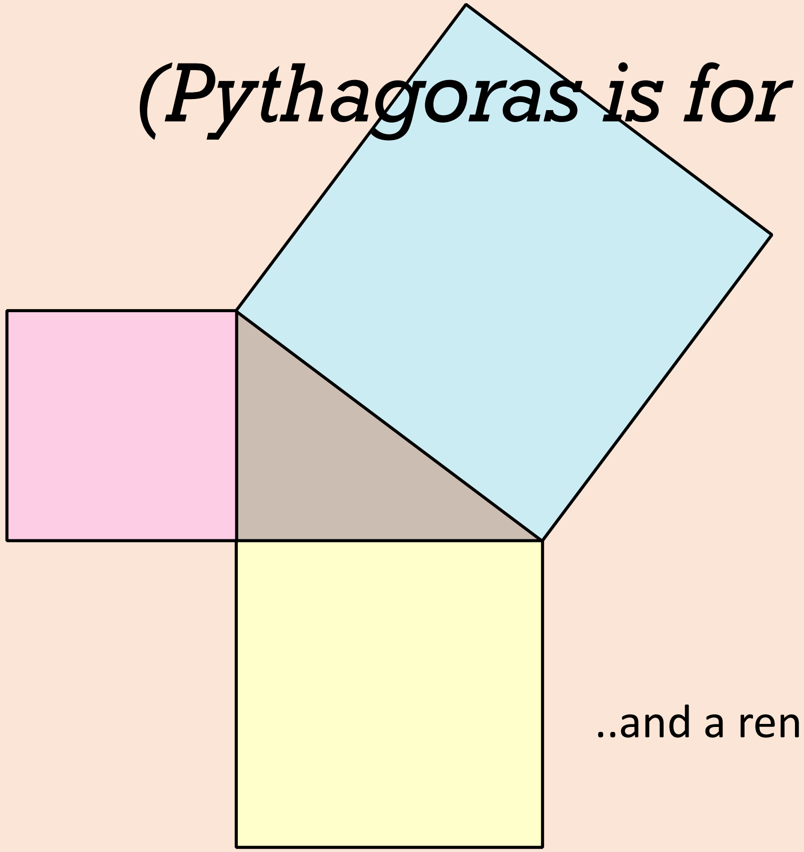
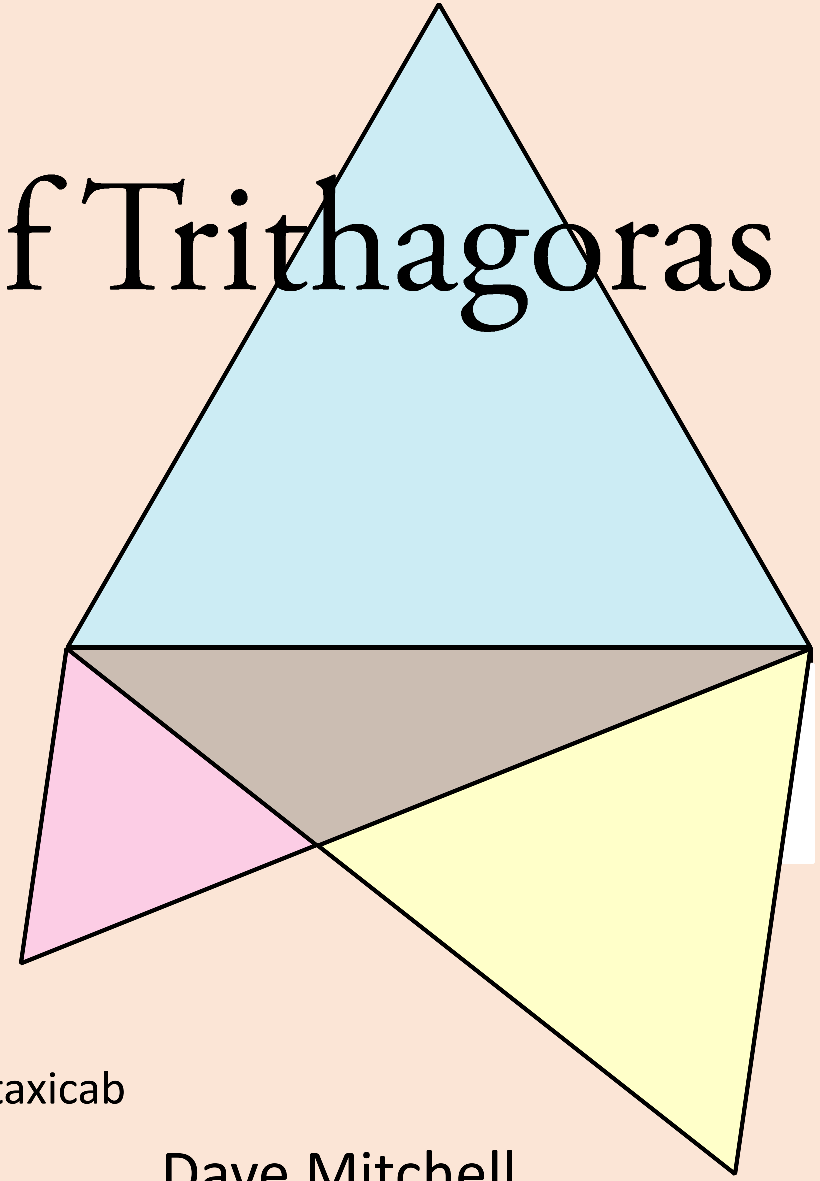
The Thereom of Trithagoras

(Pythagoras is for SQUARES)



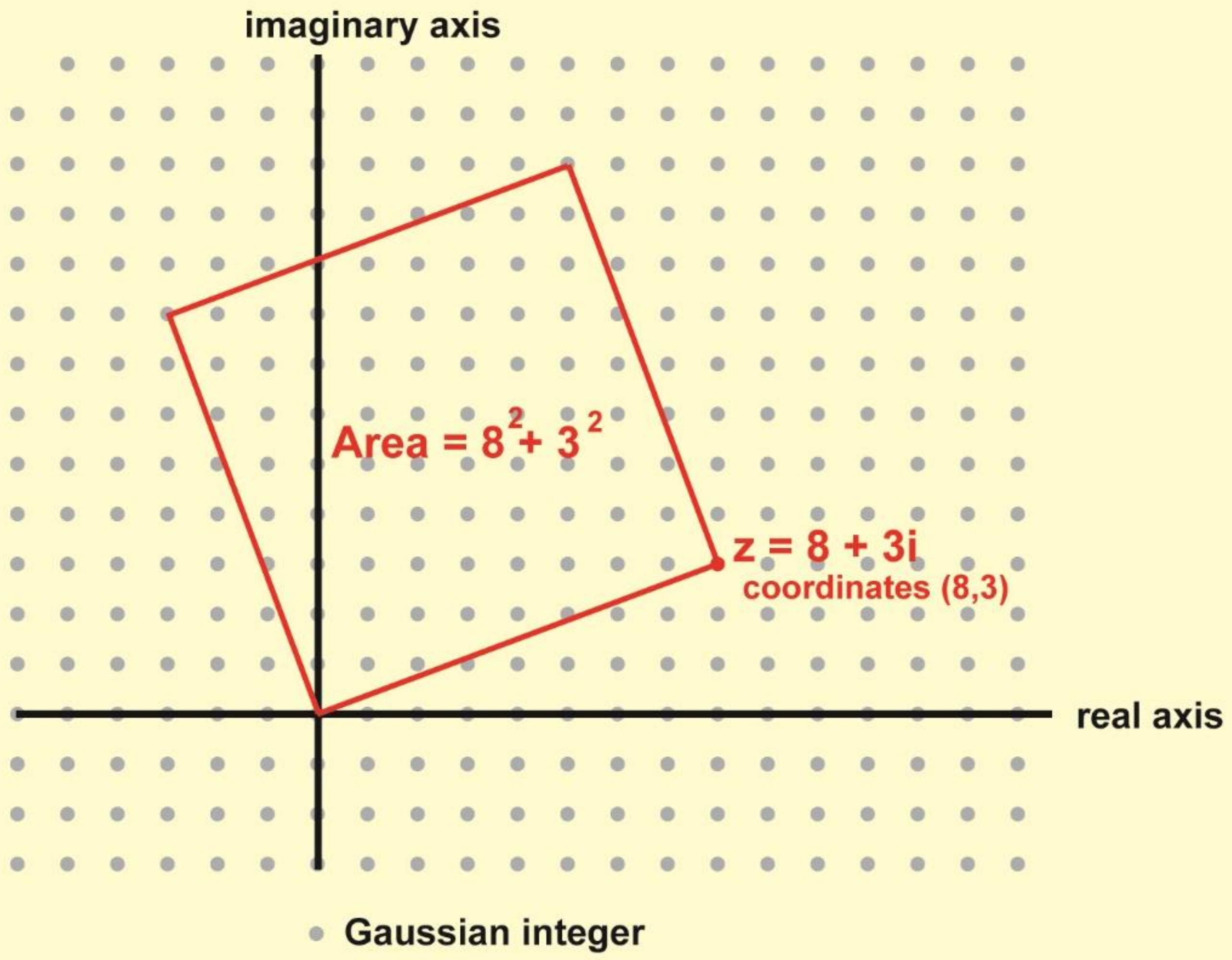


..and a rendezvous with Hardy's taxicab

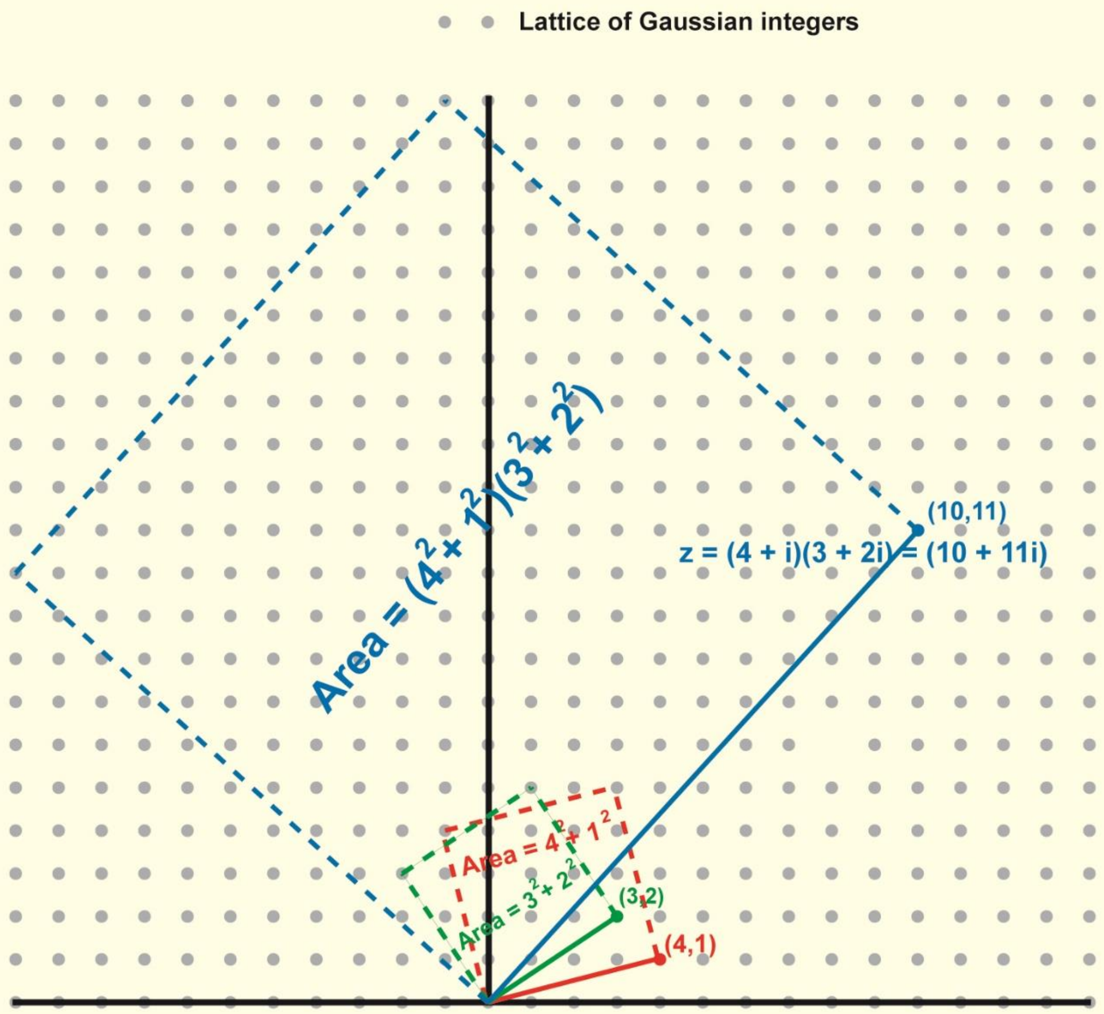
Dave Mitchell

MathsJam 2017

Constructing a square on the unit square lattice on the Argand diagram



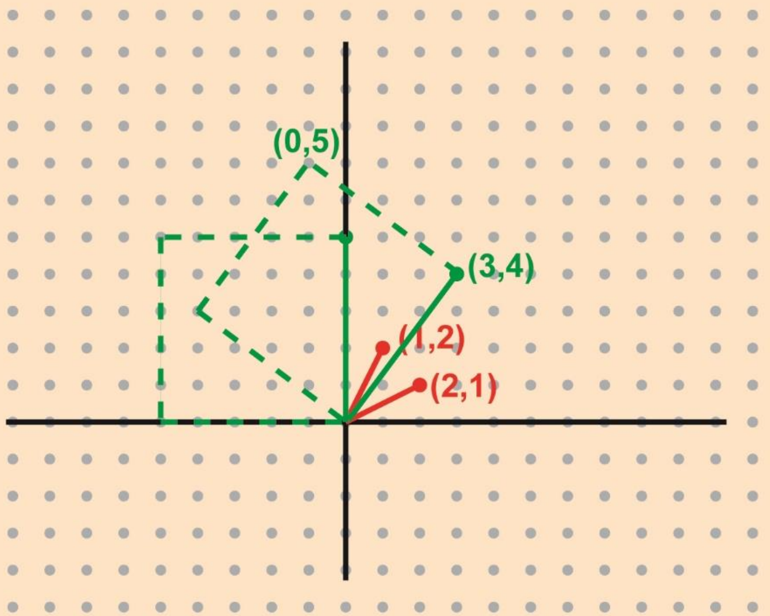
Multiplying complex numbers and the squares they specify



in general, $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$

P.S. is there another digital complex number specifying a square of area equal to the blue square?

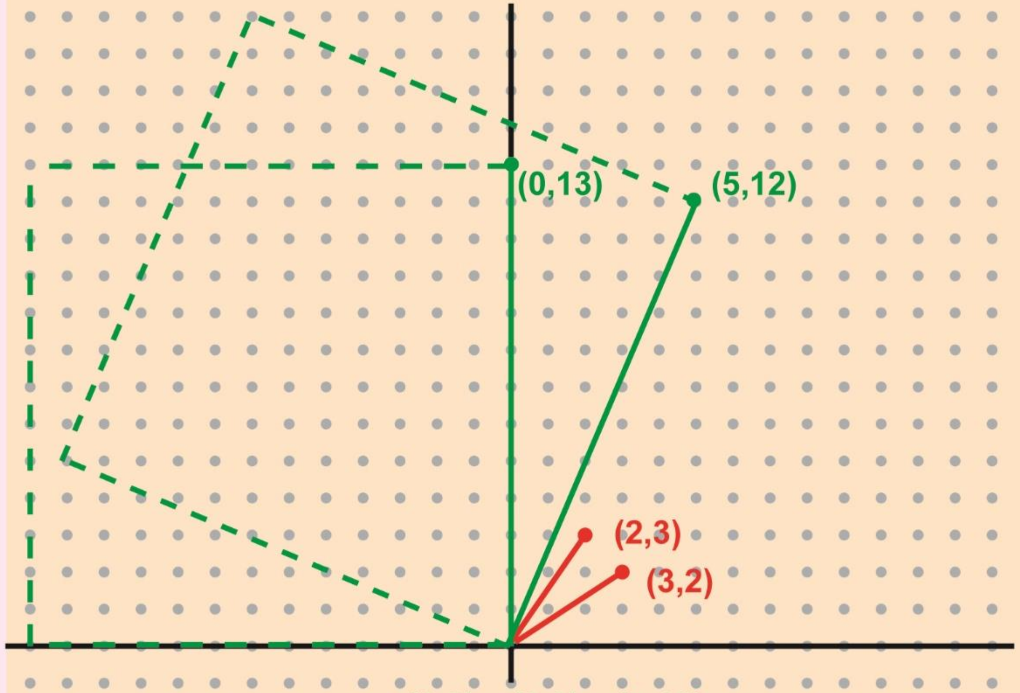
Finding Pythagorean triples by squaring and "reflection squaring" Gaussian integers



(2,1) x (2,1) = (3,4)

(2,1) x (1,2) = (0,5)

Area of square = $(2^2 + 1^2)^2 = 3^2 + 4^2 = 5^2 = 25$



(3,2) x (3,2) = (5,12)

(3,2) x (2,3) = (0,13)

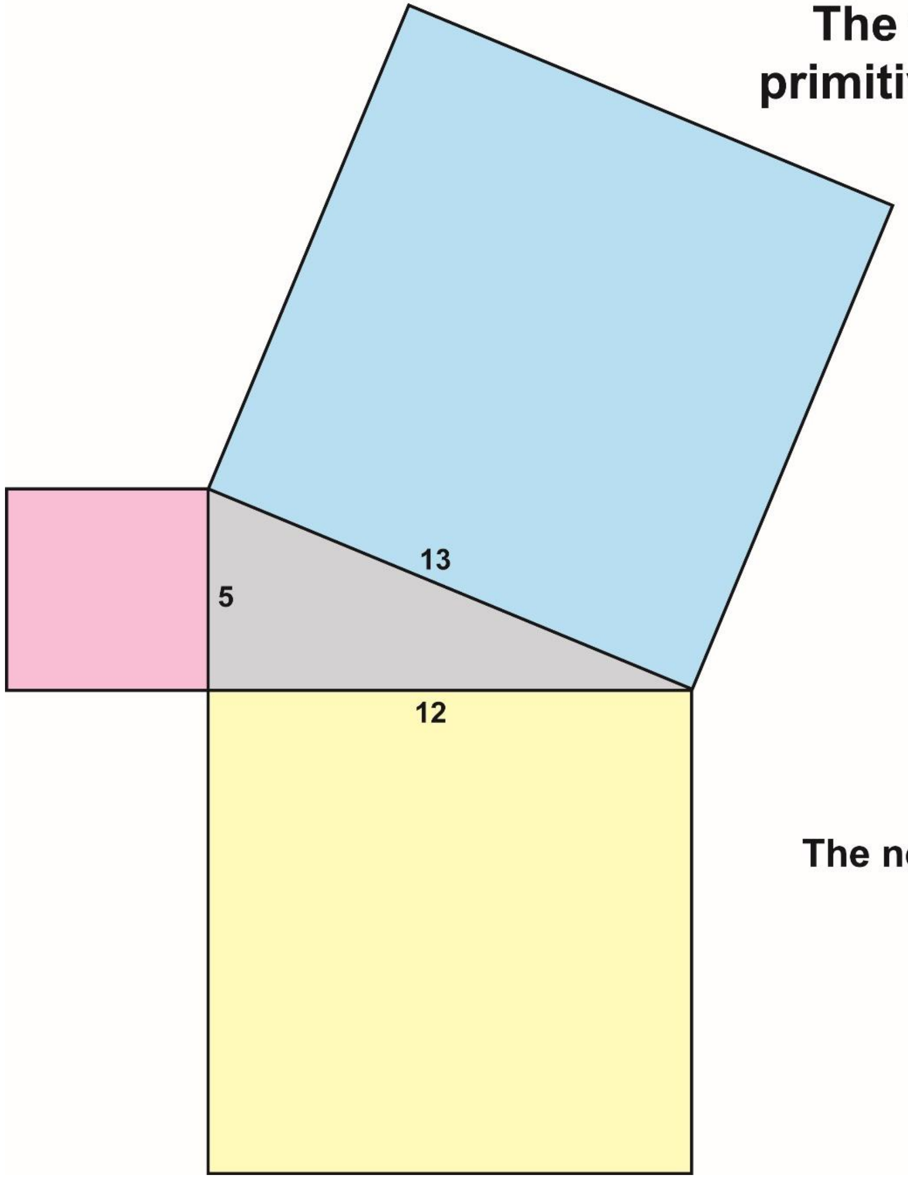
Area of square = $(3^2 + 2^2)^2 = 5^2 + 12^2 = 13^2 = 169$

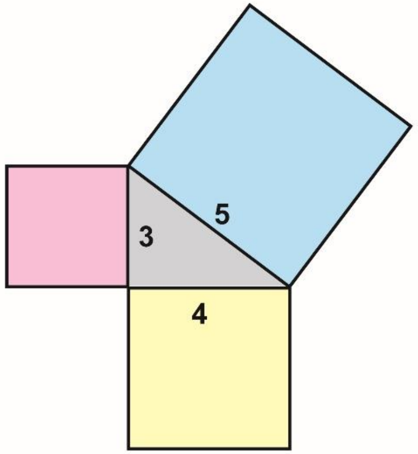
likewise (3,1) x (3,1) = (8,6), (3,1) x (1,3) = (0,10) , (4,1) x (4,1) = (15,8), (4,1) x (1,4) = (0,17) and so on for all triples

Note, in general : $(a + bi)(a + bi) = (a^2 - b^2) + 2abi$

$(a + bi)(b + ai) = (a^2 + b^2)i$

The two lowest-order primitive Pythagorean Triples

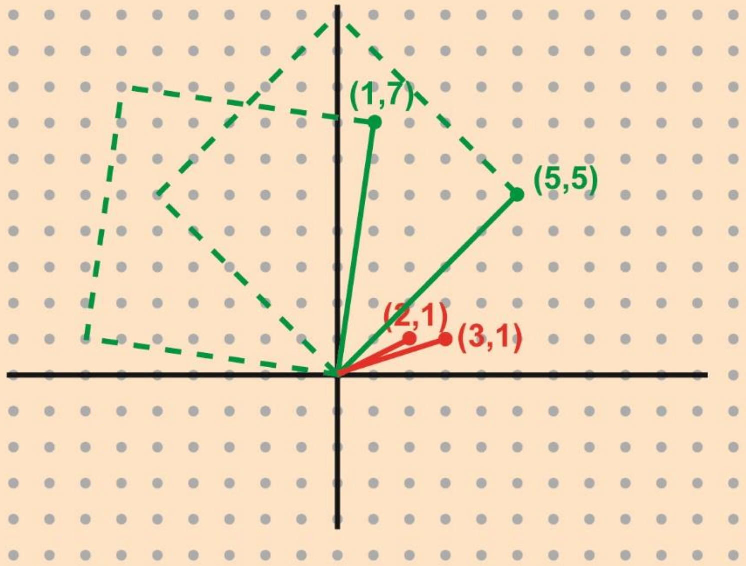




The next Pythagorean Triple is 8, 15, 17

Finding Pythagorean quads by multiplying distinct Gaussian integers

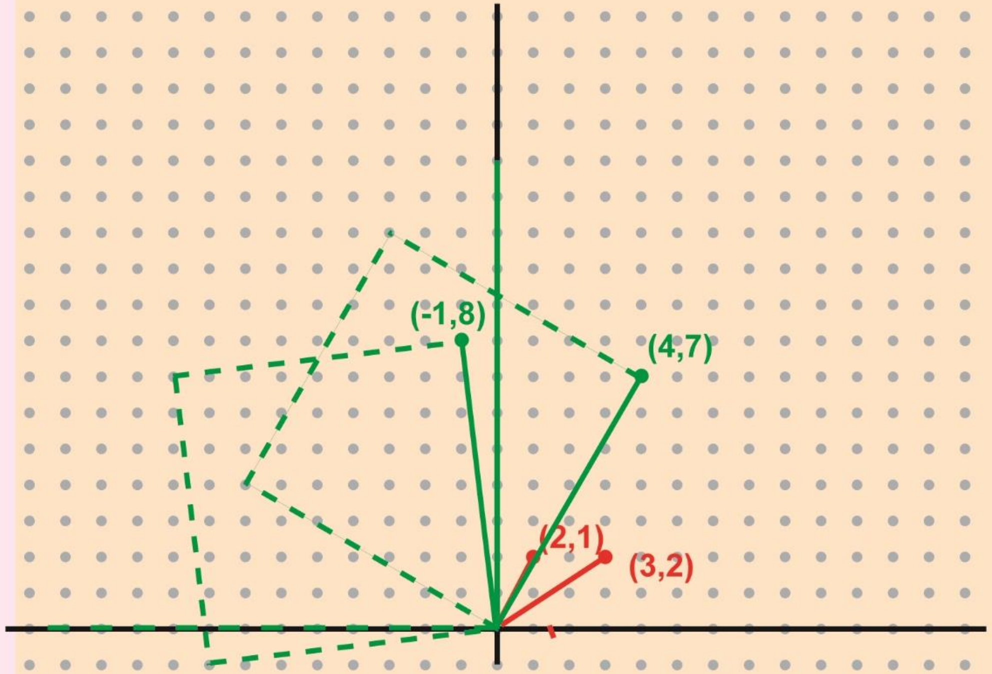
5, 5 ; 1, 7



(3,1) x (2,1) = (5,5) (3,1) x (1,2) = (1,7)

Area of square = $(3^2 + 1^2) \times (2^2 + 1^2)$
$= (5^2 + 5^2) = (1^2 + 7^2) = 50$

4, 7; 1, 8

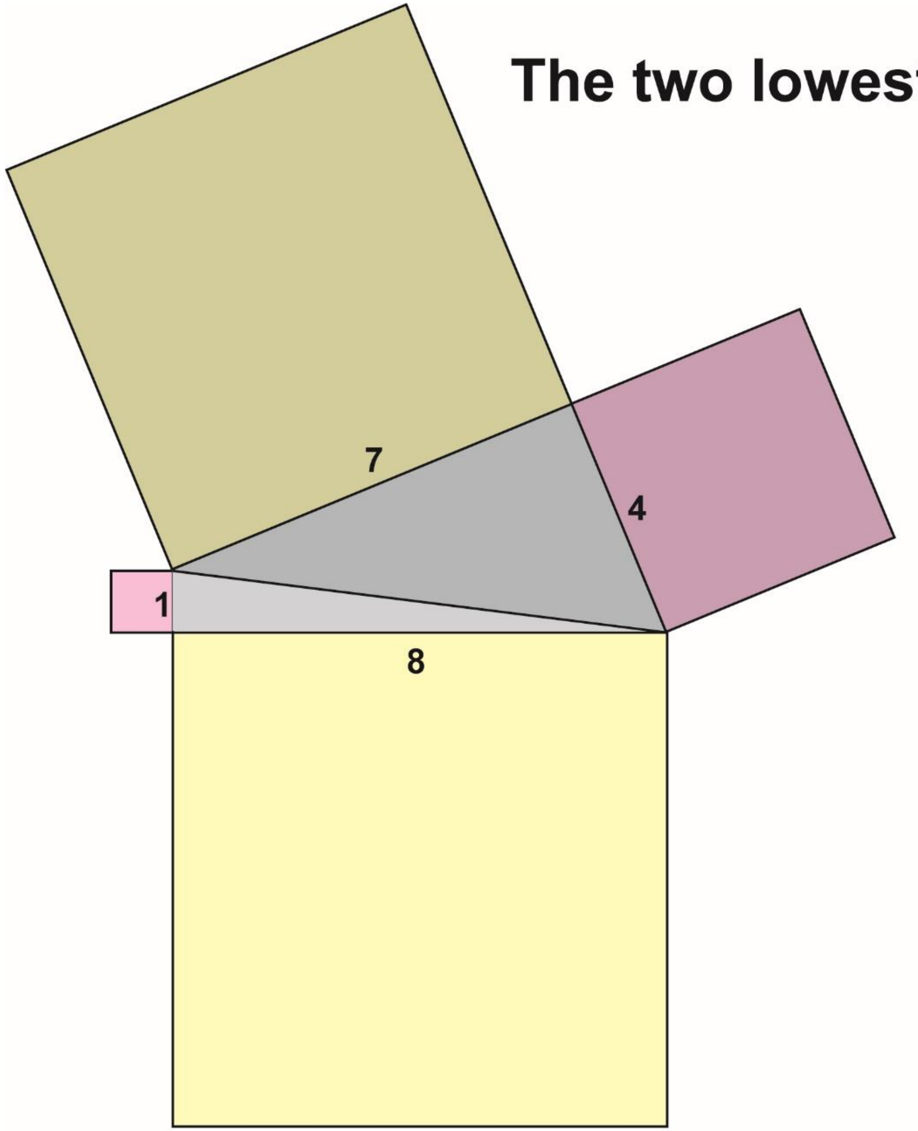


(3,2) x (2,1) = (4,7) (3,2) x (1,2) = (-1,8)

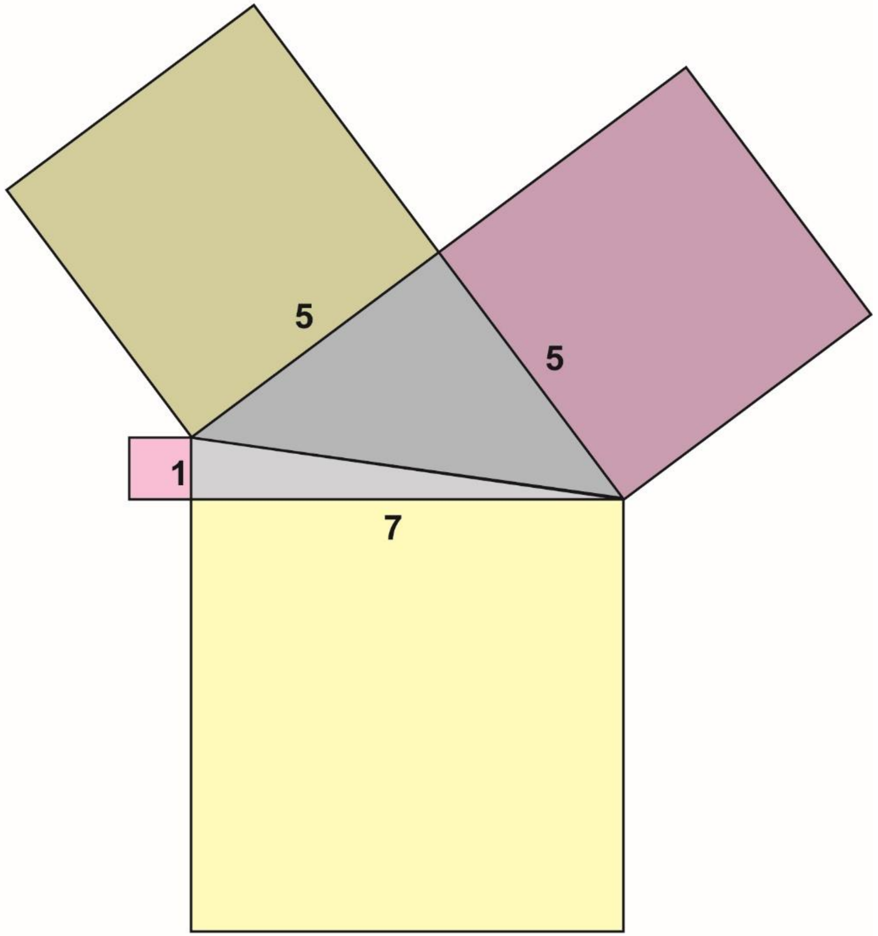
area of square = $(3^2 + 2^2) \times (2^2 + 1^2)$
$= (4^2 + 7^2) = (1^2 + 8^2) = 65$

Note, in general : (a + bi)x(c + di) = ac-bd + (ad+bc)i
or in co-ordinates: (a,b) x (c,d) = (ac-bd , ad + bc)

The two lowest-order Pythagorean Quads



$1^2 + 8^2 = 7^2 + 4^2 = 65 = 5x13$

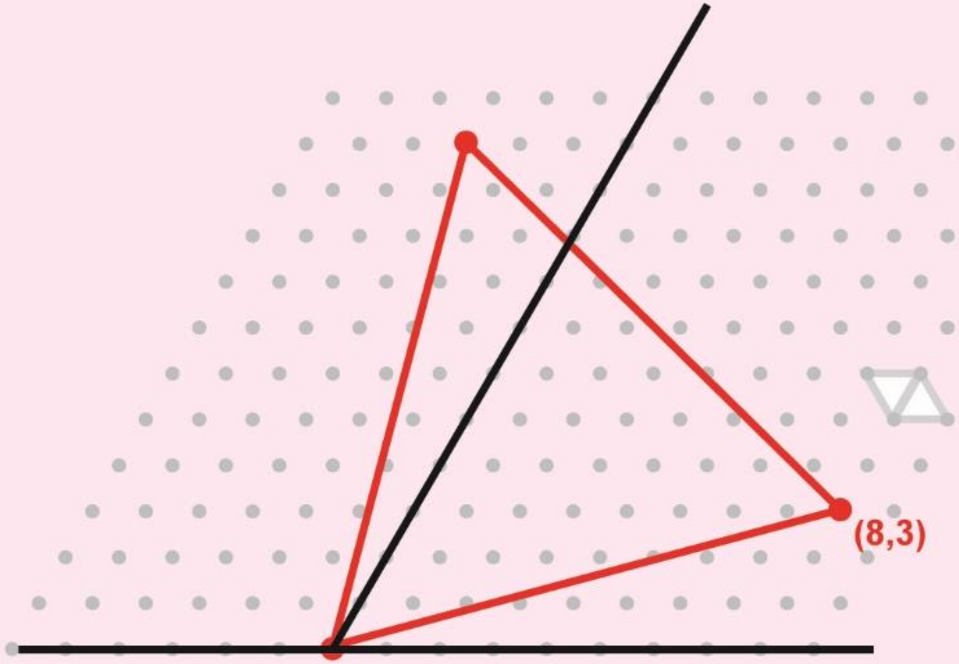


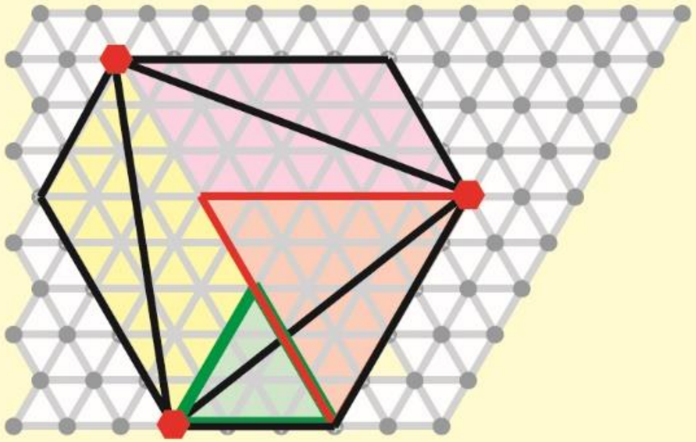
$1^2 + 7^2 = 5^2 + 5^2 = 25 = 5x5$

The lowest-order "sextuple": $15^2 + 10^2 = 17^2 + 6^2 = 18^2 + 1^2 = 325 = 5x5x13$

The lowest-order "octuple": $24^2 + 23^2 = 31^2 + 12^2 = 32^2 + 9^2 = 33^2 + 4^2 = 1105 = 5x13x17$

Constructing an equilateral triangle on the unit triangular lattice on the Argand Diagram



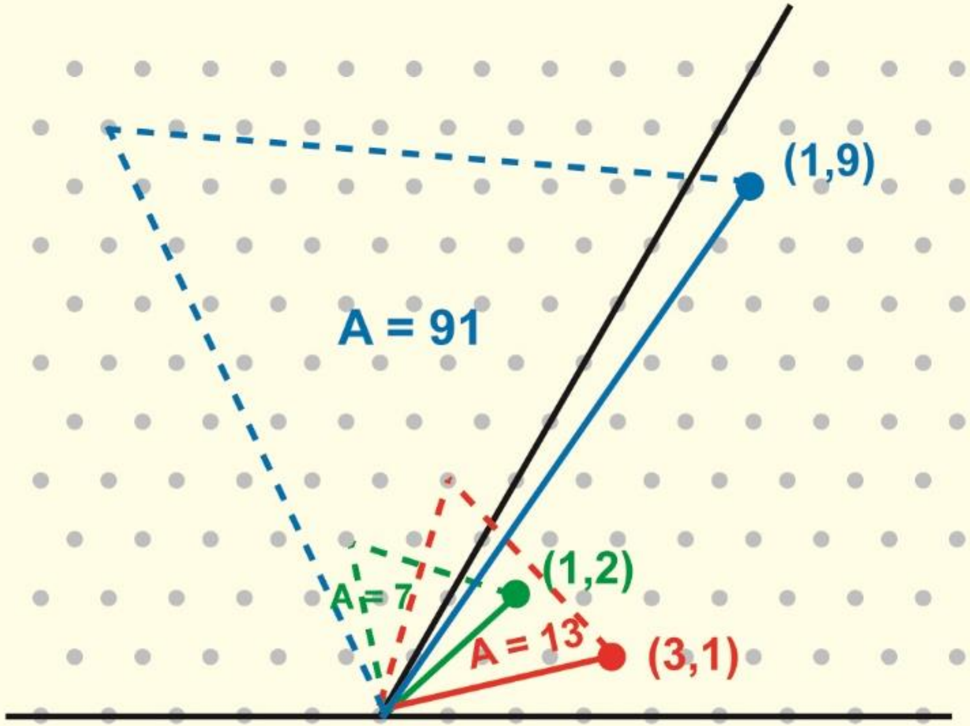


(geometrical proof of formula)

Area of equilateral triangle = $8^2 + 8 \times 3 + 3^2 = 97$ unit triangles

in general, $A = a^2 + ab + b^2$

Multiplying complex numbers, integral on the triangular lattice and so multiplying the triangles they specify.



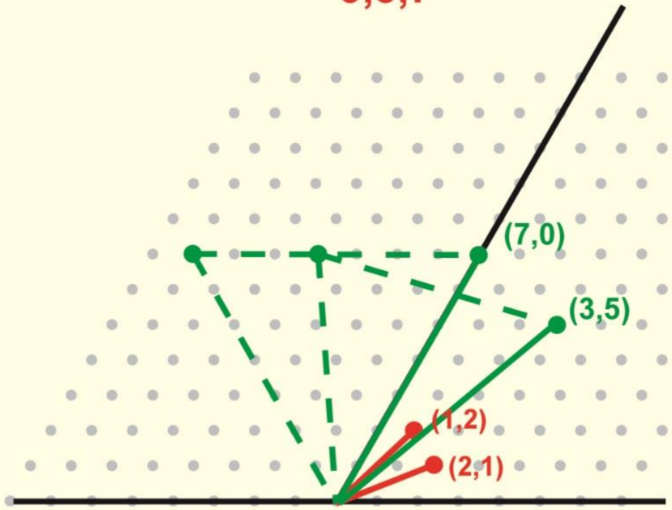
(3,1) x (1,2) = (1,9)

Triangle areas: $(3^2 + 3\times1 + 1^2)(1^2 + 1\times2 + 2^2) = 1^2 + 1\times9 + 9^2$

in general, $(a+bi)(c+di) = ac-bd + (ad + bc + bd)i$

Finding Trithagorean Triples by squaring and "reflection squaring" Eisenstein/Loeschian integers

3,5,7

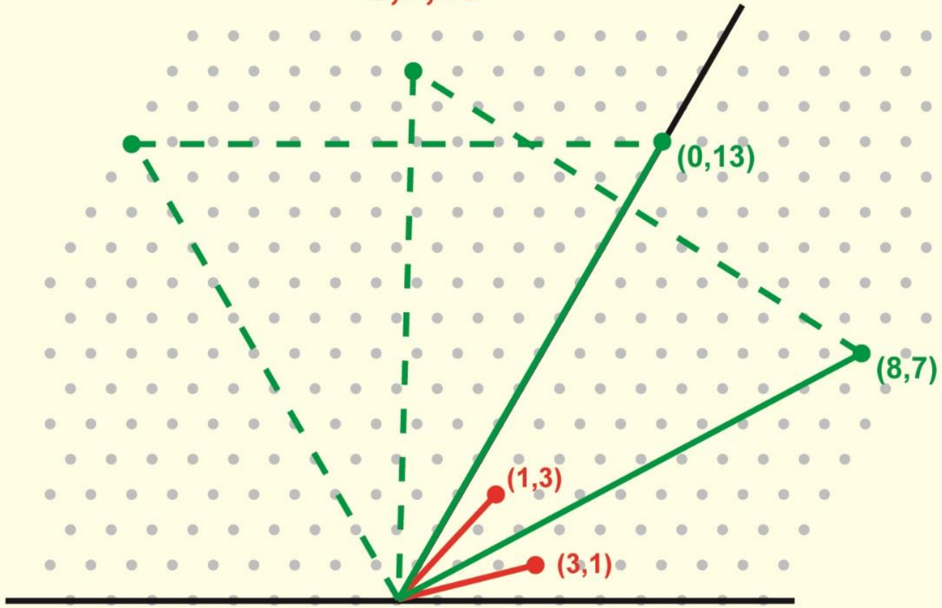


(2,1) x (2,1) = (3,5)

(2,1) x (1,2) = (0,7)

Area of triangle = $(2^2 + 2\text{x}1 + 1^2)^2 = 3^2 + 3\text{x}5 + 5^2 = 7^2 = 49$

8,7,13



(3,1) x (3,1) = (8,7)

(3,1) x (1,3) = (0,13)

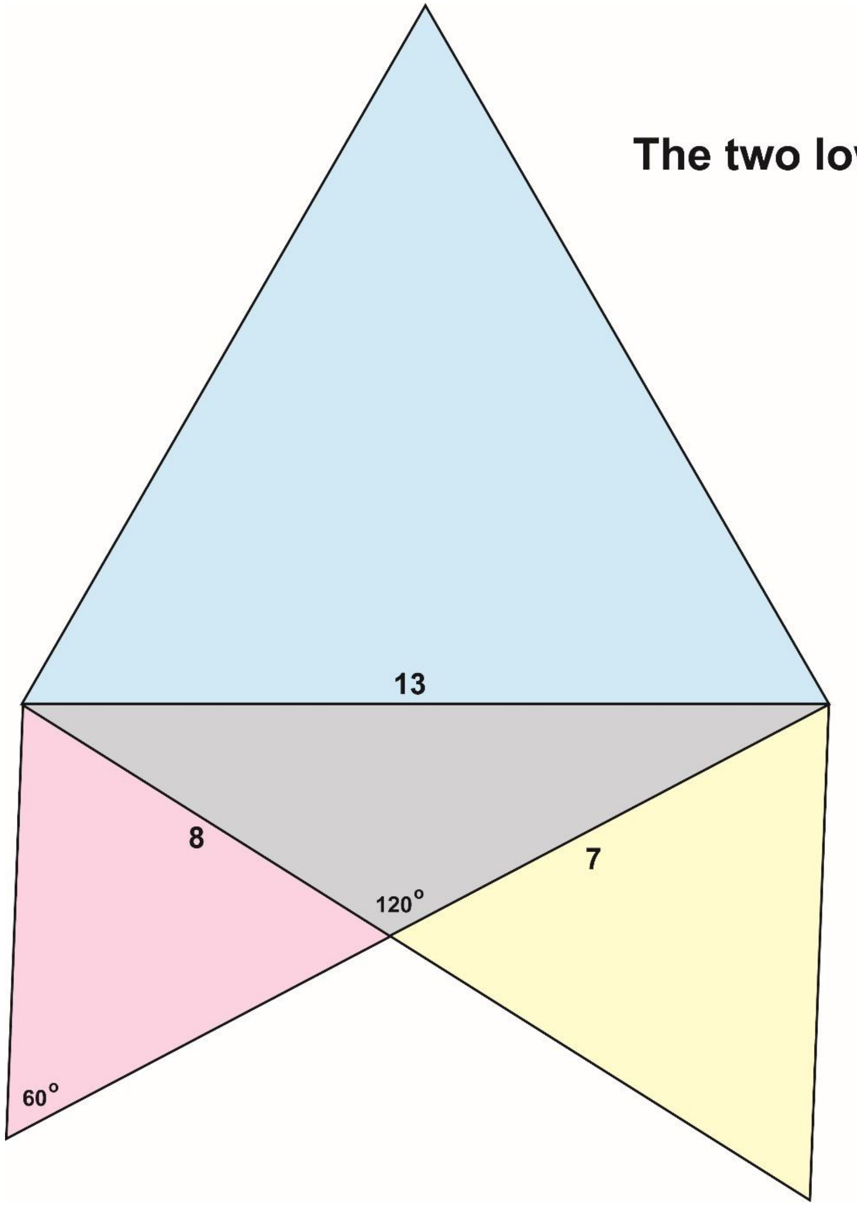
Area of triangle = $(3^2 + 3\text{x}1 + 1^2)^2 = 8^2 + 8\text{x}7 + 7^2 = 13^2 = 169$

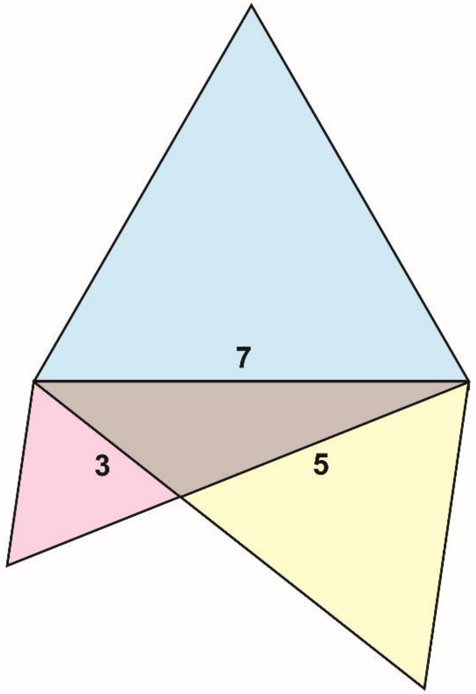
likewise (3,2) x (3,2) = (5,16) , (4,1) x (4,1) = (9,15)

(3,2) x (2,3) = (0,19) (4,1) x (1,4) = (0,21)

5,16,19 **9,15,21**

The two lowest-order Trithagorean triples

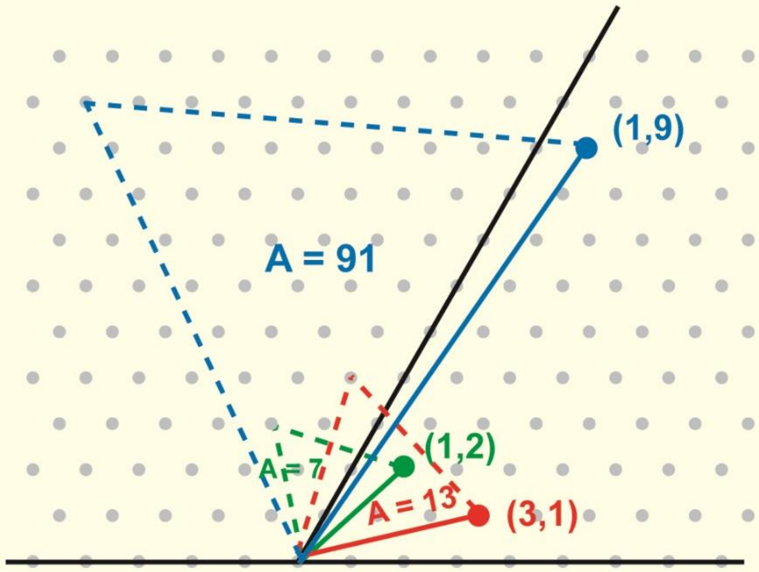




The next Trithagorean triple is 16, 5, 19.

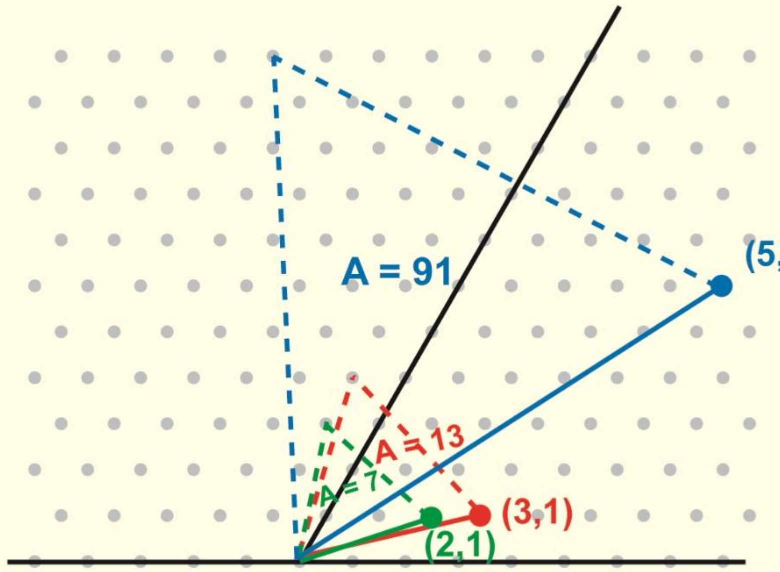
Finding Trithagorean Quads by multiplying distinct Eisenstein/Loeschian integers

1, 9; 5, 6



(3,1) x (1,2) = (1,9)

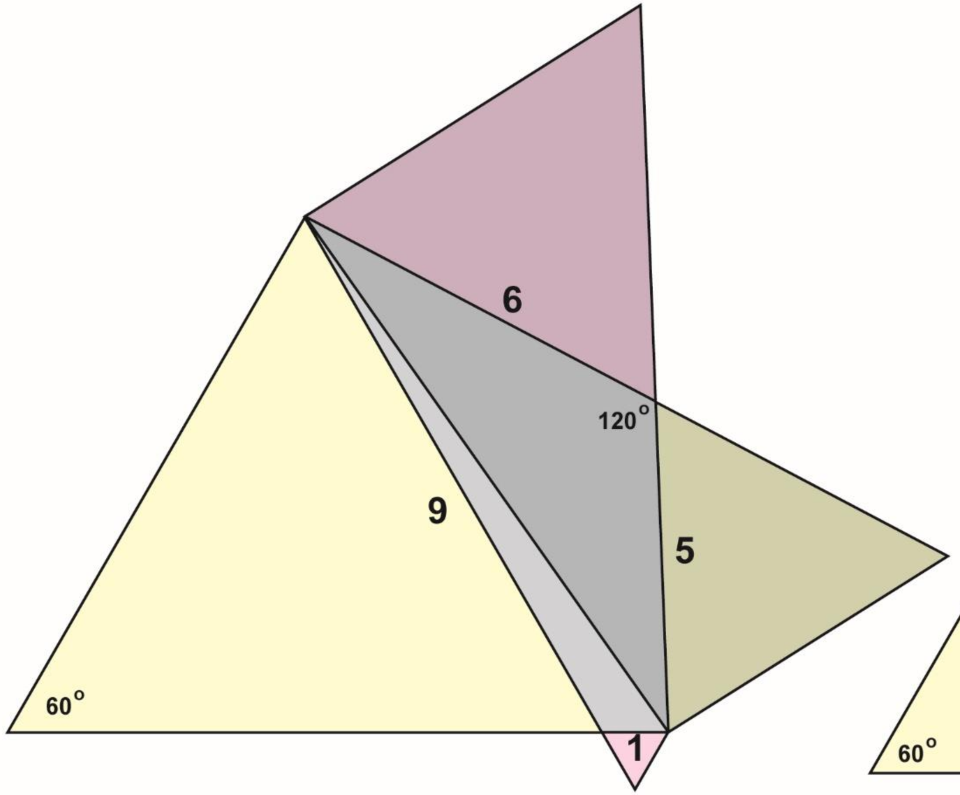
Triangle areas: $(3^2 + 3x1 + 1^2) \times (1^2 + 1x2 + 2^2) = 1^2 + 1x9 + {}^{2}9$

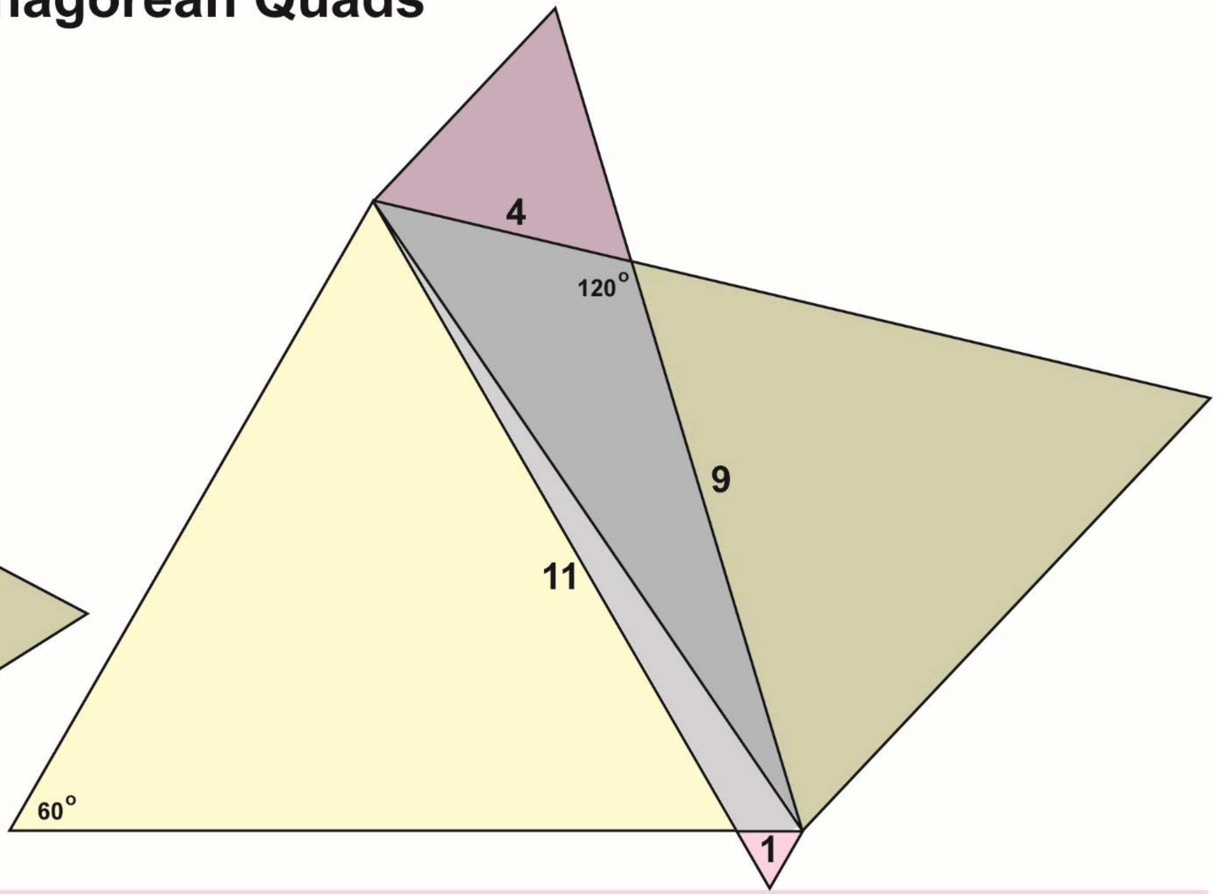


(3,1) x (2,1) = (5,6)

Triangle areas: $(3^2 + 3x1 + 1^2) \times (2^2 + 2x1 + 1^2) = 5^2 + 5x6 + 6^2$

The two lowest-order Trithagorean Quads





The lowest-order Trithagorean Sextuple: $(17^2 + 17\text{x}12 + 12^2) = (21^2 + 21\text{x}7 + 7^2) = (23^2 + 23\text{x}4 + 4^2) = 637 = 7\text{x}7\text{x}13$

The lowest-order Trithagorean Octuple: $(25^2 + 25\text{x}23 + 23^2) = (32^2 + 32\text{x}15 + 15^2) = (37^2 + 37\text{x}8 + 8^2) = (40^2 + 40\text{x}3 + 3^2) = 1729 = 7\text{x}13\text{x}19$

1729 is the beloved Hardy-Ramanujan Taxicab Number, the smallest sum of two positive cubes in two ways = $1^3 + 12^3 = 9^3 + !0^3$

https://latticelabyrinths.wordpress.com/ http://www.scarthinbooks.com/