Irrational Bases

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POSITIVE BASE $\beta (\beta > 1)$ RÉNYI β-EXPANSIONS (1957)

If $\beta > 1$, Rényi's so-called greedy β-expansion of x in the half-open interval [0,1) is the particular representation in base β

$$.x_1 x_2 x_3 \cdots$$

where

$$i = 1$$
$$x_i = \lfloor \beta x \rfloor$$
$$x = \beta x - x_i$$
$$i = i+1$$

If a representation has trailing 0s then these can be ignored and the representation is finite.

β-expansions ordered by value are in lexicographic order.

POSITIVE BASE β(β>1) RÉNYI β-EXPANSIONS EXAMPLE

What is 3 in base $\beta = 1+\phi = 2.6180339887...$?

Divide by a suitable power of β: $x = 3/\beta^2 = 0.43769$ is in the interval [0,1).

$x \mapsto x - \lfloor \beta x \rfloor$	βx	$\lfloor \beta x \rfloor$
0.43769	1.14590	1
0.14590	0.38197	0
0.38197	1	1
0		

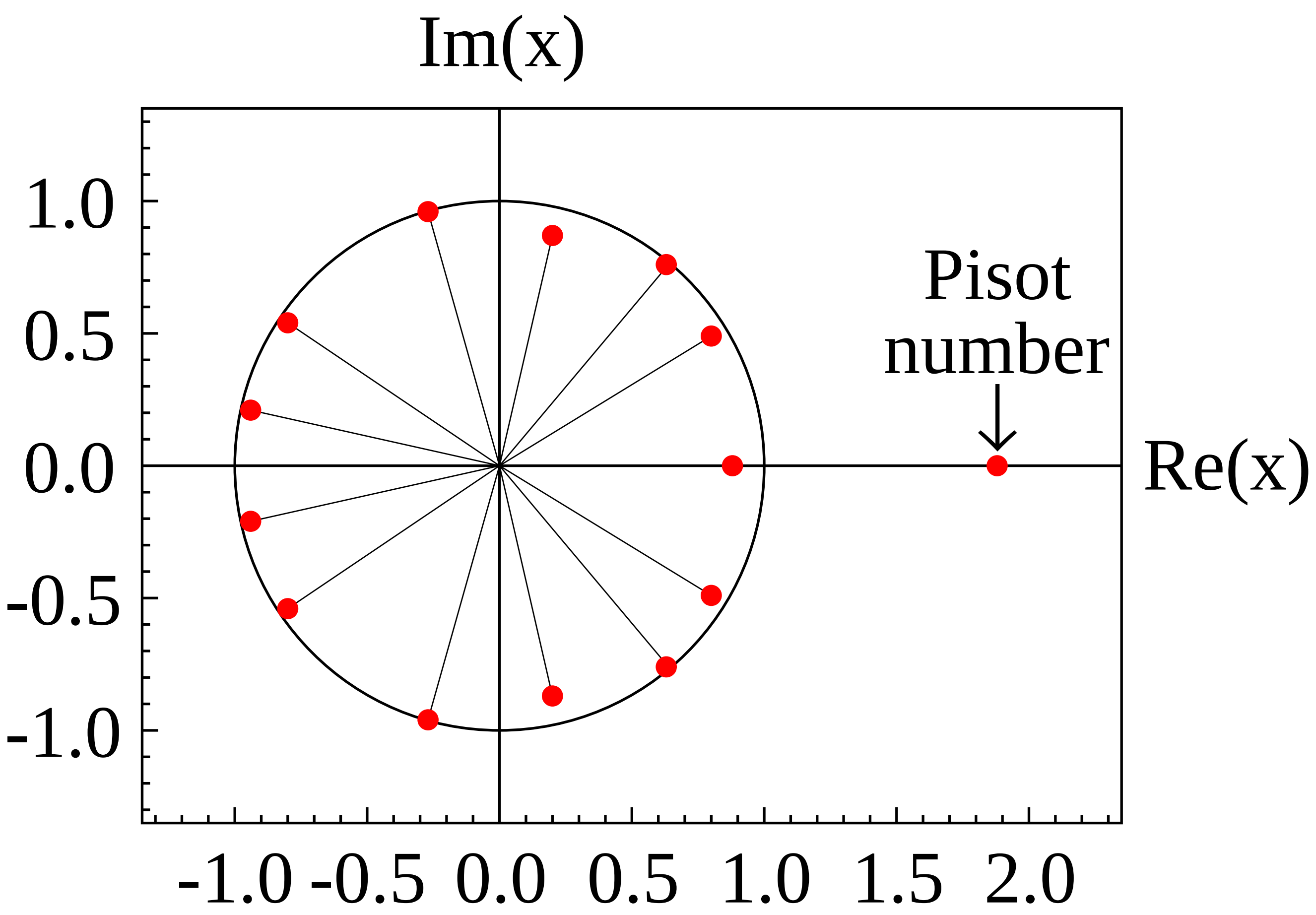
$$3_{10} = 10.1_{\beta}$$

PISOT NUMBERS

A Pisot number $\beta > 1$ is a root of a polynomial with integer coefficients and leading coefficient 1, whose other roots all have modulus less than 1.

To determine the roots of a polynomial, enter it at www.wolframalpha.com

x^14-2x^13+x^11-x^10-x^7+x^6-x^4+x^3-x+1 has two real roots, 1.8800004786... and 0.8825080140..., and 12 complex roots.



POSITIVE BASE β(β>1) FINITENESS PROPERTY (F)

If every integer can be written in a finite form in base β then β is said to possess the **finiteness Property (F)** and must be a Pisot number.

Every Pisot number of degree 1 or 2 has this finiteness property.
Here are those less than 4:

β	Minimal polynomial
1.61803 39887...	x^2-x-1
2	x-2
2.41421 35623...	x^2-2x-1
2.61803 39887...	x^2-3x+1
3	x-3
3.30272 56377...	x^2-3x-1
3.41421 35623...	x^2-4x+2
3.56155 28128...	x^2-3x-2
3.73205 08075...	x^2-4x+1
3.79128 78474...	x^2-3x-3

POSITIVE BASE $\beta (\beta > 1)$ FINITENESS PROPERTY (F)

A cubic Pisot number has the finiteness Property (F) if and only if it is a root of the minimal polynomial

x^3 - ax^2 - bx - 1

where a>=0 and -1 <= b <= a+1.

(Akiyama 2000)

<u>Example</u>

β = 2.52137 97068... with minimal polynomial x^3-3x^2+2x-2 does NOT have Property (F).
For example,
6 = 20.210(00112)^ω
has no finite representation.

NEGATIVE BASES AND ALTERNATE ORDER

A negative base allows numbers, both positive and negative, to be represented without the use of a minus sign. However, they may require more digits than with the positive-base counterpart.

The natural order for numbers in a negative base, ordered by value, is alternating lexicographic order, also called alternate order.

The lexicographic and alternate orders of two numbers are the same if and only if the leftmost digit position where the two numbers differ is associated with an even power of the base.

NEGATIVE BASE $-\beta$ ($\beta>1$) ITO-SADAHIRO ($-\beta$)-EXPANSIONS (2009)

Ito & Sadahiro's ($-\beta$)-expansion of x in the interval $\left[-\frac{\beta}{\beta+1}, \frac{1}{\beta+1}\right)$ is the particular representation in base $-\beta$

$$.x_1x_2x_3\cdots$$

where

$i=1$

$$x_i = \left\lfloor \frac{\beta}{\beta+1} - \beta x \right\rfloor$$

$$x = -\beta x - x_i$$

$i = i+1$

NEGATIVE BASES
FINITENESS PROPERTIES

If every integer can be expressed in a finite form in base -β,

- using only Ito-Sadahiro (-β)-expansions, then β is said to possess the **finiteness Property (-F)** (standard terminology);

- without being restricted to using Ito-Sadahiro (-β)-expansions, then β is said to possess the general **finiteness Property (-G)** (my terminology).

If β has Property (-F) then, by definition, it also has Property (-G).

NEGATIVE BASES
FINITENESS PROPERTY (-F)

Minimal polynomial	**β has Property (-F) if and only if**
Masaková, Pelantová & Vavrá (2010) arxiv.org/abs/1002.1009	
$x^2 - mx + n$	$m >= n+2 >= 3$
Vavrá (2014) arxiv.org/abs/1404.1274	
$x^3 - mx^2 - mx - n$ $m >= n >= 1$	$n = m$
Krčmáriková, Steiner & Vavrá (2017) arxiv.org/abs/1701.04609	
$x^3 - ax^2 + bx - c$, $\|c\| = 1$	$c = 1, -1 <= b < a$, and $\|a\| + \|b\| >= 2$
Other cubics	To be determined

NEGATIVE BASES FINITENESS CONJECTURE 1

If the Ito-Sadahiro representation of -1 is finite then β has Property (-F). Is the conjecture true, or do any of the following provide counterexamples?

β	Minimal polynomial	Base -β -1
2.83928 67552...	x^3-4x^2+4x-2	12.1222
2.89328 91963...	x^3-3x^2+x-2	12.12
3.19582 91963...	x^3-3x^2-2	12.02
3.26953 08420...	x^3-4x^2+3x-2	13.32
3.45667 83430...	x^3-4x^2+3x-2	13.212
3.59867 45078...	x^3-4x^2+2x-2	13.22
3.68909 53236...	x^3-3x^2-2x-2	13.102
3.83117 72072...	x^3-5x^2+5x-3	13.1232
3.87512 97941...	x^3-4x^2+x-2	13.12
3.91963 95658...	x^3-5x^2+5x-3	13.1323
3.93946 50585...	x^3-4x^2+x-3	13.13

NEGATIVE BASES FINITENESS CONJECTURE 2

If -1 can be expressed in a finite form in base -β then β has Property (-G). Is the conjecture true, or do any of the following β, none of which have Property (-F), provide counterexamples?

β	Minimal polynomial	Base -β: -1	Base -β: 0
1.32471 79572...	x^3-x-1	0.10001	110001
1.46557 12318...	x^3-x^2-1	0.101	1101
2.26953 08420...	x^3-x^2-2x-2	0.201020	120102
2.35930 40859...	x^3-2x^2-2	0.202	1202
2.52137 97068...	x^3-3x^2+2x-2	12.101012	*
3.22069 28199...	x^3-2x^2-3x-3	0.30203	130203
3.27901 87861...	x^3-3x^2-3	0.303	1303
3.37442 37632...	x^3-4x^2+3x-3	13.202321120103	*
3.67799 34833...	x^3-4x^2+2x-3	13.23	*
3.74734 65403...	x^3-3x^2-2x-3	13.113	*

*: no zero-value string with any non-zero digits

ALTERNATIVE REPRESENTATIONS IN NEGATIVE BASES

As an example, in base -1.46557..., a better alternative to
3 = 11111.00(110001)^ω (Ito-Sadahiro) is
3 = 101.011 where

- the representation is finite, if there is such a representation;
- the integer part is as short as possible;
- no subset of digits, not all zero and not necessarily contiguous, has the value zero.

The alternative representation of any number multiplied by the base should have the same digit sequence as the original number.

Is there an algorithm to find a representation which satisfies the above properties, or the highest such representation in alternate order?

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