From a twitter puzzle to Landau's 4th problem in <5min

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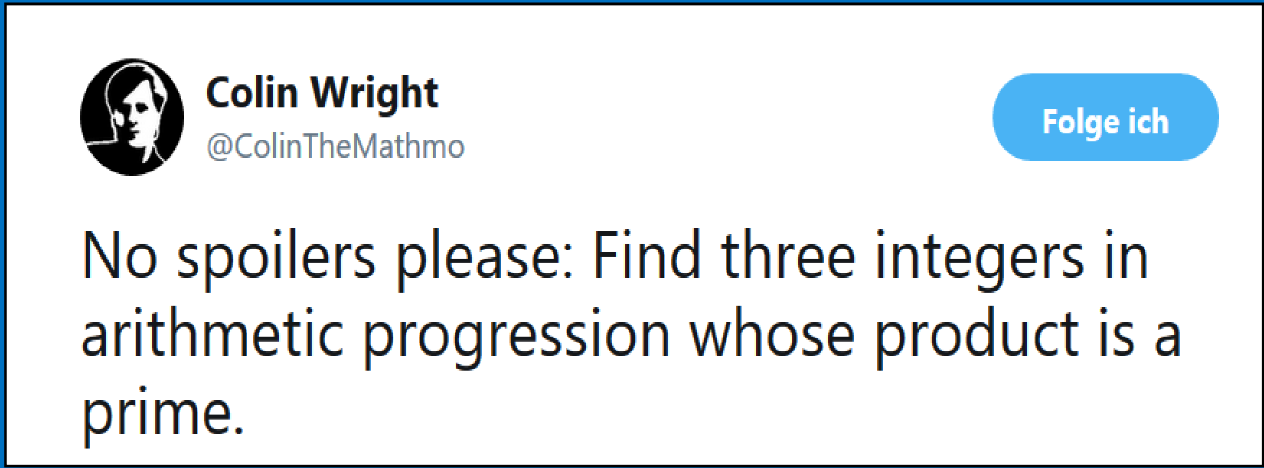


Slightly more precise:

Find an arithmetic progression $a, b, c \ \in \mathbb{Z} : abc$ is prime in $\mathbb{N}$

Looks impossible, but then think again (after this talk).

Modifying the riddle:

Find an arithmetic progression $a, b, c \in \mathbb{Q} : abc$ is prime in $\mathbb{N}$

How many solutions?

Modifying the riddle 2: going complex

Find an arithmetic progression $a, b, c \in \mathbb{G} = \mathbb{Z} + \mathrm{i}\mathbb{Z} : abc$ is prime in $\mathbb{N}$

Gaussians for benefit and profit

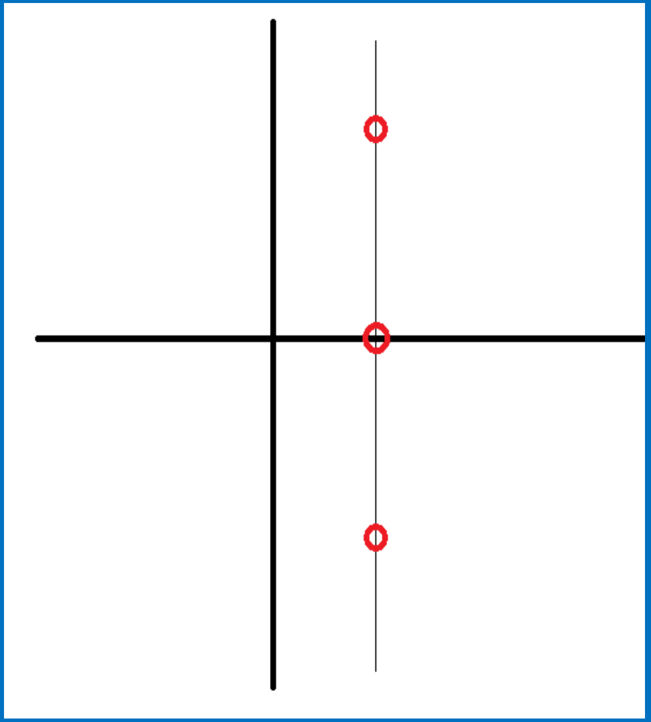
Primes in $\mathbb{G}$ relate to the primes in $\mathbb{N}$ (prime classification theorem).

Finding a general form of the solutions (and the proof that these are all solutions) is easier than it might look.

The only candidates which are not part of the original riddle are of the form

$a = 1 - \mathrm{i}n$
$b = 1$
$c = 1 + \mathrm{i}n$

The product gives $abc = n^2 + 1$.



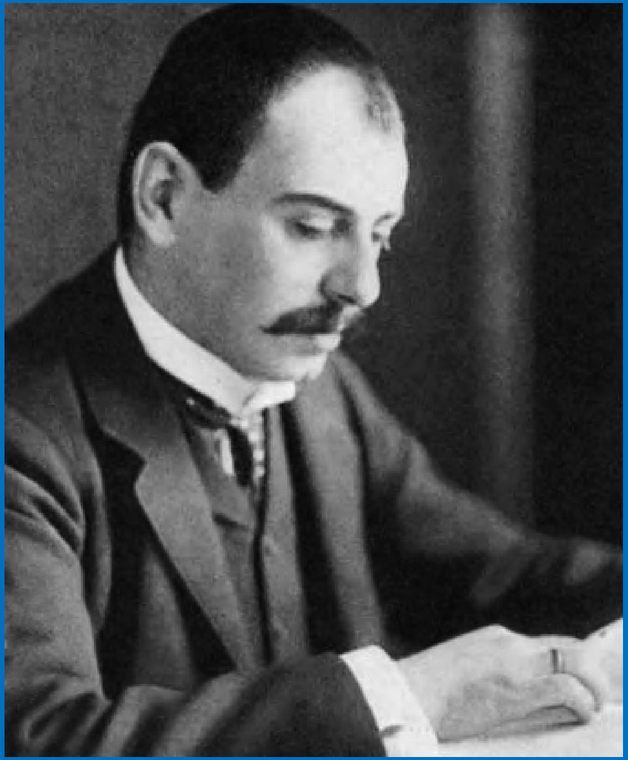
How many of the $n^2 + 1$ are prime in $\mathbb{N}$?

Lots of examples
$(2^2 + 1 = 5,\ 4^2 + 1 = 17,\ 6^2 + 1 = 37)$

Edmund Landau presented 4 problems in 1912 (IMC in Cambridge)

1: **Twin prime conjecture**

Status: still unsolved



2: **Goldbach conjecture**

Status: Still unsolved

3. **Legendre conjecture:** there is always a prime between n^2 and $(n+1)^2$.

Status: still unsolved

4. **no name conjecture:** there are infinitely many primes of the form n^2+1.

Status: still unsolved

Next puzzle

Find two non-isomorphic groups A and B with the following homomorphisms

An injection $A \hookrightarrow B$

A surjection $A \twoheadrightarrow B$

An injection $B \hookrightarrow A$

A surjection $B \twoheadrightarrow A$

What about TOP?

Better next puzzle

Train fares *usually/supposedly* define a metric $d(x, y)$ on the set of all train stations

(1) Positivity: $d(x, y) > 0, \; d(x, x) = 0$

(2) Symmetry: $d(x, y) = d(y, x)$

(3) Triangle inequality

British trains don’t satisfy (3), especially when the 3 stations are on the same line (just why??). How can you visualise this?