

# Why Knot?

R Low

MathsJam Gathering 2017

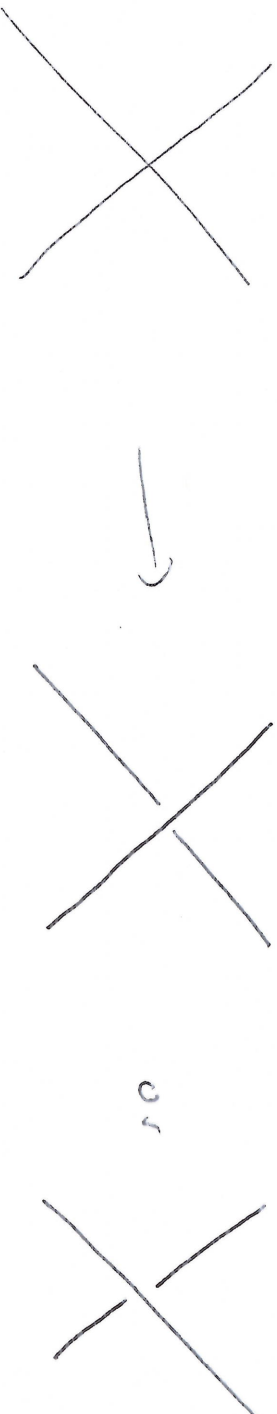
12th November, 2017

We all “know” that you can’t tie a knot in a length of string without letting go of the ends.

But how do we know this?

Is it really impossible, or just really complicated, and we haven’t figured out the right manipulations?

We can represent any knot by looking at its projection, i.e. at the shadow it casts on a flat sheet, and decorating the crossing points by breaking the line segment that passes underneath:



**Assertion 1:** Any knot has a projection in which at most two segments meet at any point, and they cross over there. (i.e. if we start with a projection in which more than two segments meeting at a point, we can deform it slightly so that this no longer happens.)

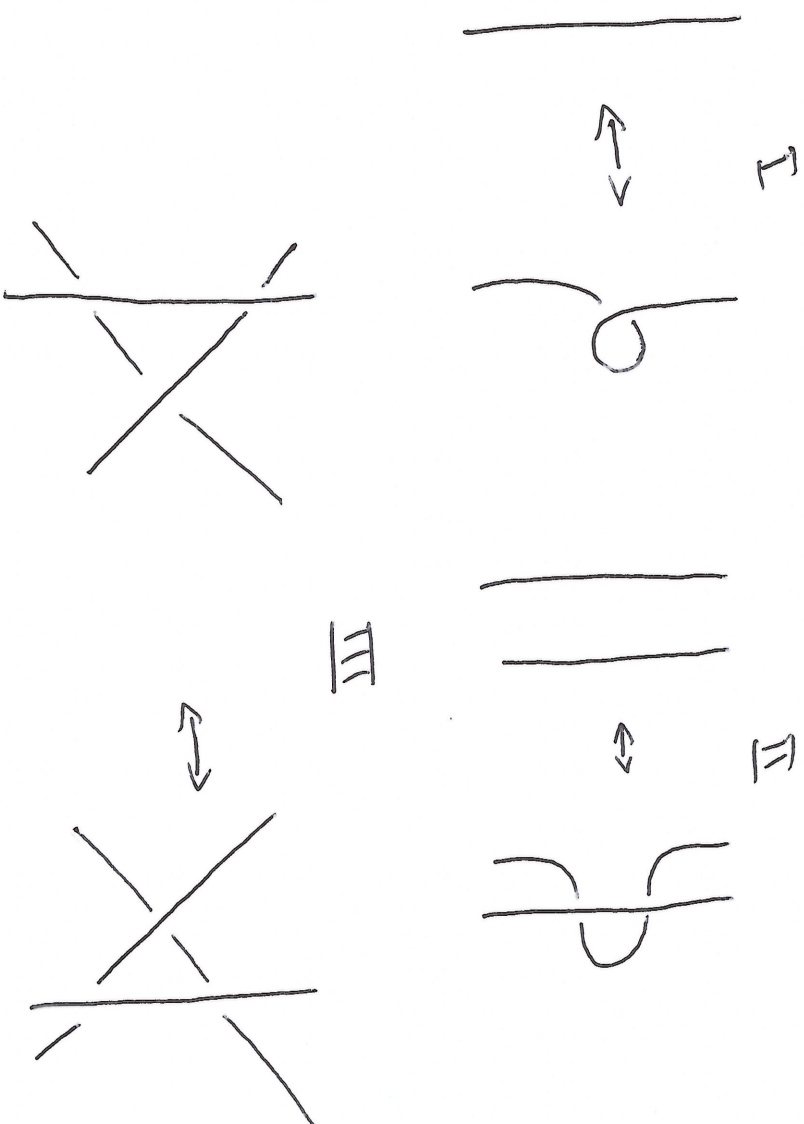


That's a little misleading. I missed out several pictures where the projection wasn't really well-behaved: there had to be an intermediate part like



So the question is, what are the "special moments" that we have to consider?

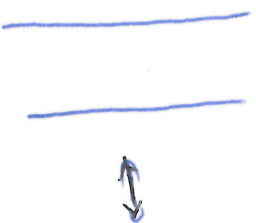
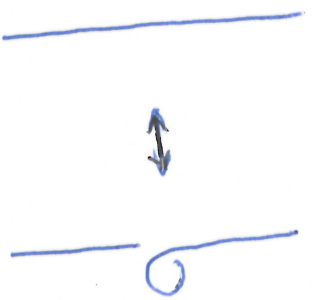
Assertion 2: All manipulations can be carried out by a sequence of the three Reidemeister moves:



Next, I call a knot projection tri-coloured if the segments are coloured using three different colours, such that at any intersection, either all three colours appear, or just one does. Obviously, the unknot (projection just a circle) cannot be tri-coloured.

Then if a knot projection is tri-coloured, and we apply a Reidemeister move, then it remains tri-coloured.

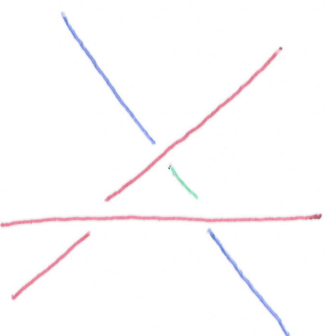
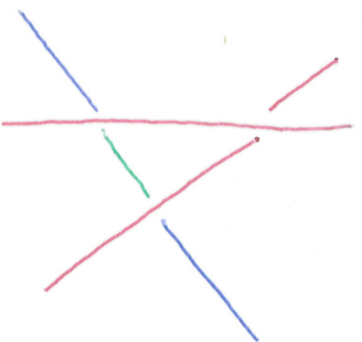
The first two cases are easy



or



The third type of move requires a bit more work. The case where all the segments appearing in the move have the same colour doesn't need any consideration. Here's one involving more colours. (The others are similar, and I don't have time to go through them. Homework.)



Ta-dai

