Counting caterpillars

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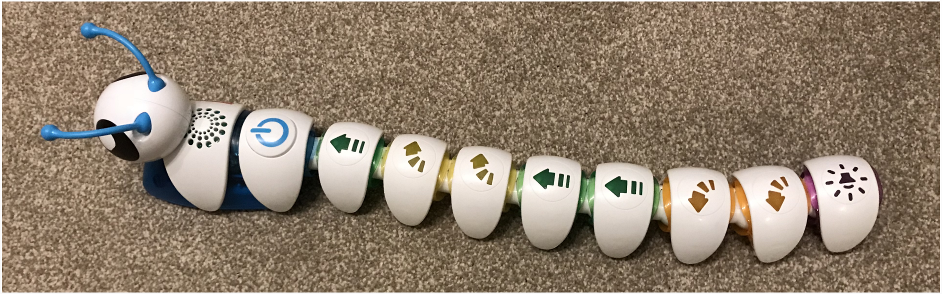
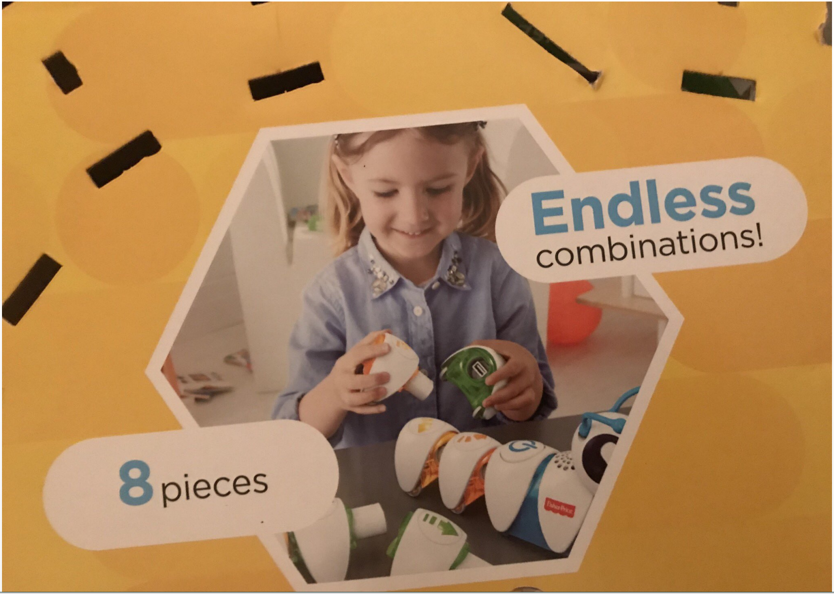
A new toy



An unbelievable claim



Enter combinatorics

- It isn't hard to establish an upper bound somewhat less than ∞.
- If all eight segments were different, the number of ways of arranging these would yield $8!$ different caterpillars.

Including shorter caterpillars

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  - we choose k segments from the 8 possible segments $\left(\text{which is } \binom{8}{k}\right)$

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- so the number of possible sequences would be

$$\binom{8}{k} k! = \frac{8!}{(8-k)!}.$$

Notice that for $k = 8$, this reduces to $8!$, as you might expect.

Putting it together

- The number of possible sequences using any number from 1 to 8 segments would be the sum of this arrangement for all possible lengths, i.e.

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- N.B. $109{,}600 < \infty$.
- For this problem, though, it's definitely too large.

Complicating factors

- The caterpillar has three identical segments which instruct it to move forwards, two to turn left, two to turn right and one to stop and play music.
- For the avoidance of doubt, some other features and restrictions:
  - there are eight positions that can hold segments, the head is always present;
  - any number of segments from 1–8 may be used, and not 0;
  - the order of segments matters, because it is a sequence of commands;
  - segments are connected (by USB), so there can be no gaps in a sequence – if a position is unfilled, the caterpillar ends at the preceding segment.

Notation

- Forwards: F;
- Left: L;
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- Music: M.

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- Forwards: F;
- Left: L;
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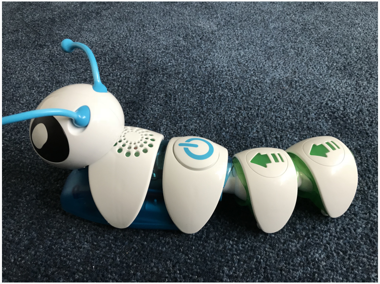
- Music: M.
- Denote a sequence by a string from left to right (the head is to the left of the string).
- e.g. $FFFM$, $FFMF$, $FLRFLRFM$, LRL, F, and so on.

Over-counting

- Previously, we were over-counting.
- For example, we counted *FF* and *FF* as different caterpillars, even though they are functionally the same.



Generating functions

- There is a neat trick in combinatorics for counting in this kind of circumstance – generating functions.
- For the three F segments, we can use the function $1 + f + f^2 + f^3$ to represent these.
- For the two R segments, use $1 + r + r^2$.
- For the two L segments, use $1 + l + l^2$.
- For the one M use $1 + m$.

Generating functions

- To find out how many different ways there are of selecting from these segments, expand

$$(1+f+f^2+f^3)(1+r+r^2)(1+l+l^2)(1+m).$$

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$$(1 + f + f^2 + f^3)(1 + r + r^2)(1 + l + l^2)(1 + m).$$

- The expansion has 72 terms.
- One is formed by multiplying the four 1s together to get 1, which represents selecting no segments. This isn't a valid caterpillar.
- So there are 71 different ways of forming caterpillars up to length 8.

One caterpillar combination

- For example, one of the terms of the expansion is $f^3 r l^2 m$.
- This represents choosing F, F, F, R, L, L and M for a caterpillar of length 7.

F, F, F, R, L, L and M

- This can be arranged 7! ways, but some of these are duplicates; how many depends which segments are involved.
  - In this case, a caterpillar involving three Fs, an R, two Ls and an M can be rearranged $\frac{7!}{3!2!} = 420$ different ways.

All 71 combinations

- Doing this calculation for all 71 combinations and summing yields 5,023 different caterpillars.
- Not endless, but certainly enough to keep us busy for a while!

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- Doing this calculation for all 71 combinations and summing yields 5,023 different caterpillars.
- Not endless, but certainly enough to keep us busy for a while!
- Oh, and the manufacturer sells add-on kits containing extra and different segments…

Thanks for listening!

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