Bernoulli Numbers Seki

$$B_m = 1 - \sum_{k=0}^{m-1} \binom{m}{k} \frac{B_k}{m-k+1}$$

$$B_0 = 1$$

Jacob Bernoulli
1655 - 1705

Takakazu Seki
1642 - 1708

T = 15s

$$S_p(n) = \sum_{k=1}^{n} k^p.$$

S_p : $1^p + 2^p + 3^p + 4^p + 5^p + 6^p + 7^p + 8^p + 9^p + \ldots + n^p$ = ?

$p = 0$: $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \ldots + 1$ = $1\,n$

One is Everything

T = 40s

$$S_p(n) = \sum_{k=1}^{n} k^p .$$

… sum of integers , squares , cubes , fourth powers …

p = 1 : $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 \ldots + n$

p = 2 : $1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 \ldots + n^2$

p = 3 : $1 + 8 + 27 + 64 + 125 + 216 + 343 + 512 + 729 \ldots + n^3$

p = 4 : $1 + 16 + 81 + 256 + 625 + 1296 + 2401 + 4096 + 6561 \ldots + n^4$

T = 60s

Bernoulli

... Atque si porrò ad altiores gradatim potestates pergere, levique negotio sequentem adornare laterculum licet :

Summae Potestatum

$$\int n = \tfrac{1}{2}nn + \tfrac{1}{2}n$$
$$\int nn = \tfrac{1}{3}n^3 + \tfrac{1}{2}nn + \tfrac{1}{6}n$$
$$\int n^3 = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}nn$$
$$\int n^4 = \tfrac{1}{5}n^5 + \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 - \tfrac{1}{30}n$$
$$\int n^5 = \tfrac{1}{6}n^6 + \tfrac{1}{2}n^5 + \tfrac{5}{12}n^4 - \tfrac{1}{12}nn$$
$$\int n^6 = \tfrac{1}{7}n^7 + \tfrac{1}{2}n^6 + \tfrac{1}{2}n^5 - \tfrac{1}{6}n^3 + \tfrac{1}{42}n$$
$$\int n^7 = \tfrac{1}{8}n^8 + \tfrac{1}{2}n^7 + \tfrac{7}{12}n^6 - \tfrac{7}{24}n^4 + \tfrac{1}{12}nn$$
$$\int n^8 = \tfrac{1}{9}n^9 + \tfrac{1}{2}n^8 + \tfrac{2}{3}n^7 - \tfrac{7}{15}n^5 + \tfrac{2}{9}n^3 - \tfrac{1}{30}n$$
$$\int n^9 = \tfrac{1}{10}n^{10} + \tfrac{1}{2}n^9 + \tfrac{3}{4}n^8 - \tfrac{7}{10}n^6 + \tfrac{1}{2}n^4 - \tfrac{1}{12}nn$$
$$\int n^{10} = \tfrac{1}{11}n^{11} + \tfrac{1}{2}n^{10} + \tfrac{5}{6}n^9 - 1n^7 + 1n^5 - \tfrac{1}{2}n^3 + \tfrac{5}{66}n$$

The 'integral' Symbol means summation in Bernoulli's notation

Quin imò qui legem progressionis inibi attentius ensperexit, eundem etiam continuare poterit absque his ratiociniorum ambabimus : Sumtâ enim c pro potestatis cujuslibet exponente, fit summa omnium n^c seu

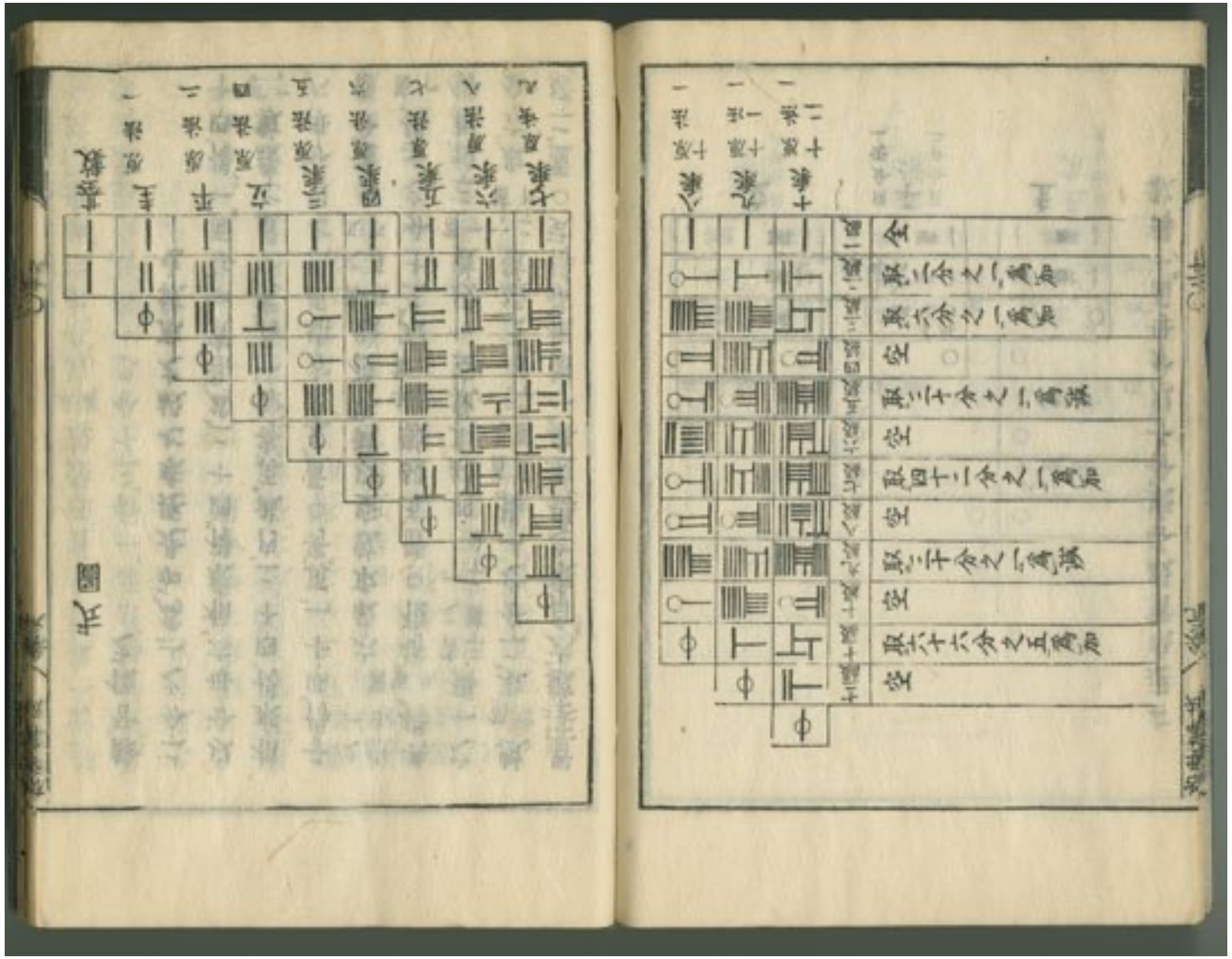
$$\int n^c = \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c\cdot c-1\cdot c-2}{2\cdot 3\cdot 4}Bn^{c-3}$$
$$+\frac{c\cdot c-1\cdot c-2\cdot c-3\cdot c-4}{2\cdot 3\cdot 4\cdot 5\cdot 6}Cn^{c-5}$$
$$+\frac{c\cdot c-1\cdot c-2\cdot c-3\cdot c-4\cdot c-5\cdot c-6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}Dn^{c-7}\ldots \text{\& ita deinceps,}$$

exponentem potestatis ipsius n continué minuendo binario, quosque perveniatur ad n vel nn. Literae capitales A, B, C, D & c. ordine denotant coëfficientes ultimorum terminorum pro $\int nn$, $\int n^4$, $\int n^6$, $\int n^8$, & c. nempe

$$A = \frac{1}{6}, B = -\frac{1}{30}, C = \frac{1}{42}, D = -\frac{1}{30}.$$

Jacob Bernoulli
Ars Conjectandi 1713

Seki



Takakazu Seki
Essentials of Mathematics 1712

T = 80s

Bernoulli

... Atque si porrò ad altiores gradatim potestates pergere, levique negotio sequentem adornare laterculum licet :

Summae Potestatum

$\int n = \frac{1}{2}nn + \frac{1}{2}n$

$\int nn = \frac{1}{3}n^3 + \frac{1}{2}nn + \frac{1}{6}n$

$\int n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}nn$

$\int n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$

$\int n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}nn$

$\int n^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$

$\int n^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}nn$

$\int n^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$

$\int n^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{1}{12}nn$

$\int n^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - 1n^7 + 1n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$

Quin imò qui legem progressionis inibi attentuis ensperexit, eundem etiam continuare poterit absque his ratiociniorum ambabimus : Sumtâ enim c pro potestatis cujuslibet exponente, fit summa omnium n^c seu

$$\int n^c = \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c \cdot c-1 \cdot c-2}{2 \cdot 3 \cdot 4}Bn^{c-3}$$

$$+ \frac{c \cdot c-1 \cdot c-2 \cdot c-3 \cdot c-4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}Cn^{c-5}$$

$$+ \frac{c \cdot c-1 \cdot c-2 \cdot c-3 \cdot c-4 \cdot c-5 \cdot c-6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}Dn^{c-7} \ldots$$ & ita deinceps,

exponentem potestatis ipsius n continué minuendo binario, quosque perveniatur ad n vel nn. Literae capitales A, B, C, D & c. ordine denotant coëfficientes ultimorum terminorum pro $\int nn$, $\int n^4$, $\int n^6$, $\int n^8$, & c. nempe

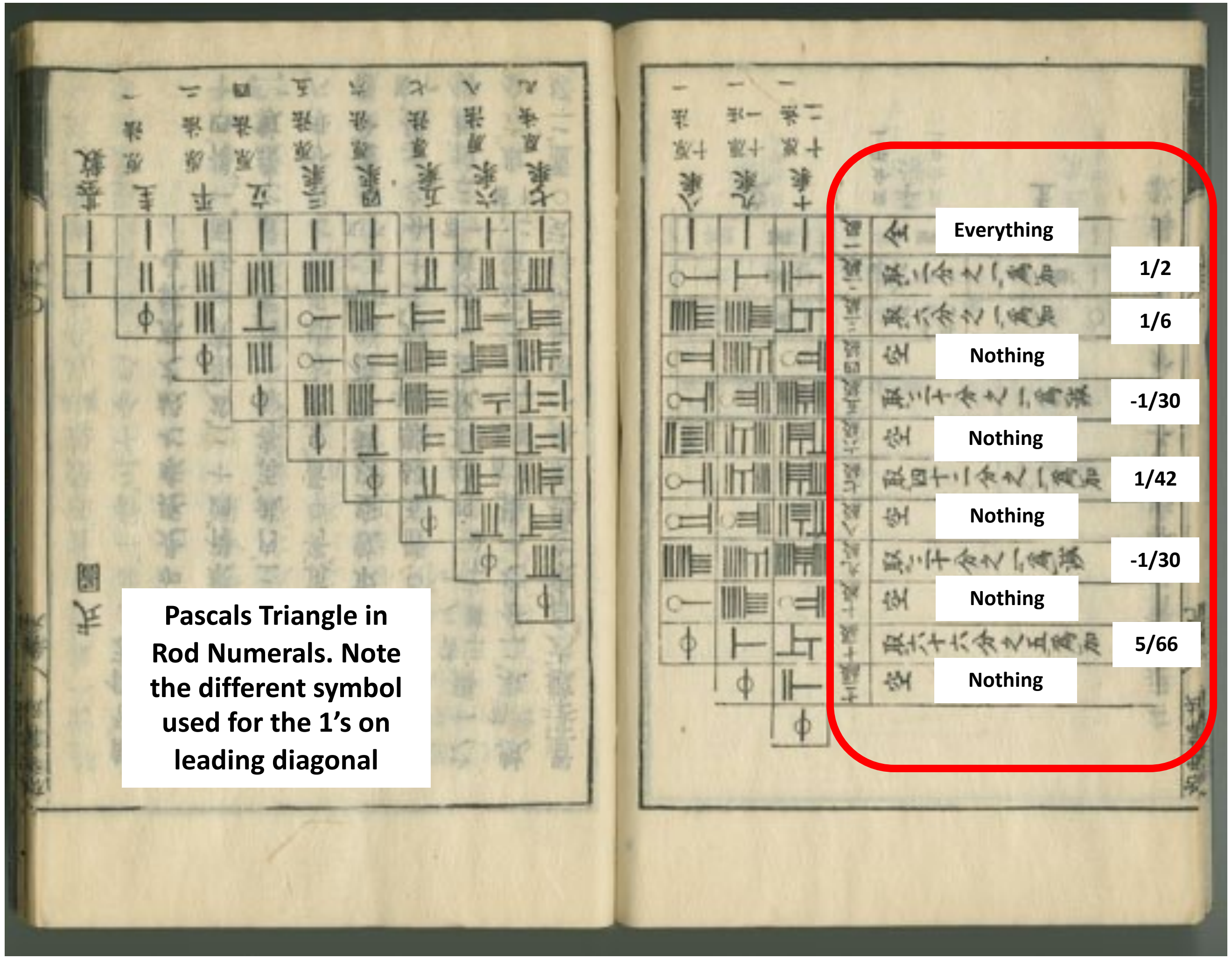
$$A = \frac{1}{6}, B = -\frac{1}{30}, C = \frac{1}{42}, D = -\frac{1}{30}.$$

Johan Faulhaber's (1580 – 1635) Formula

He Knew the first 17 cases

Jacob Bernoulli
Ars Conjectandi 1713

Seki

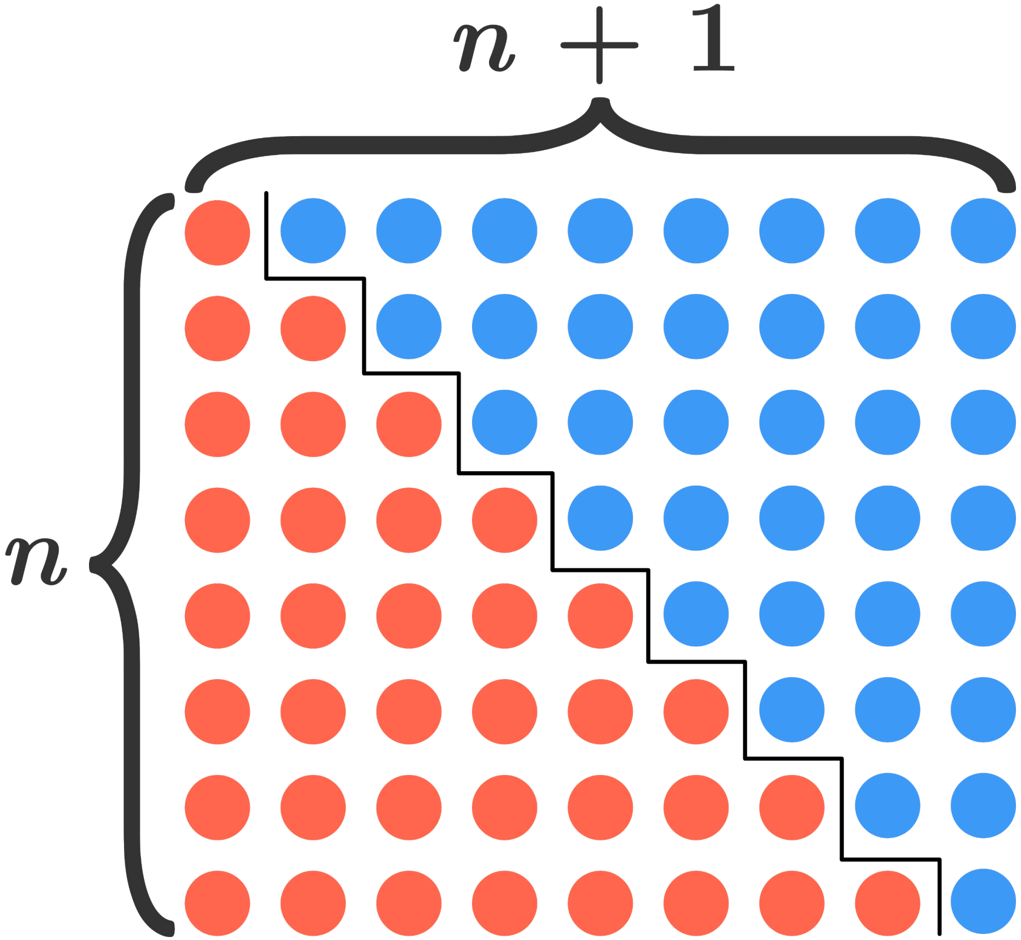


Takakazu Seki
Essentials of Mathematics 1712

T = 100s

p = 1

$$\sum = \frac{100(100+1)}{2} = \boxed{5050}$$



$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

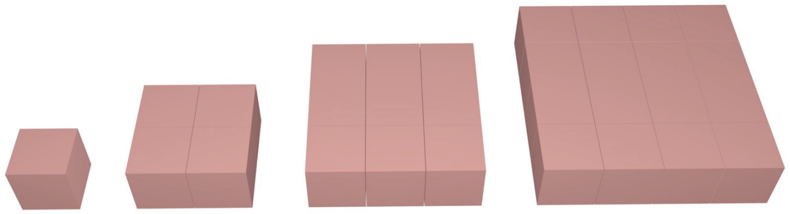
$= S_1(n) = ½\, n^2 + ½\, n$

https://brilliant.org/wiki/sum-of-n-n2-or-n3/

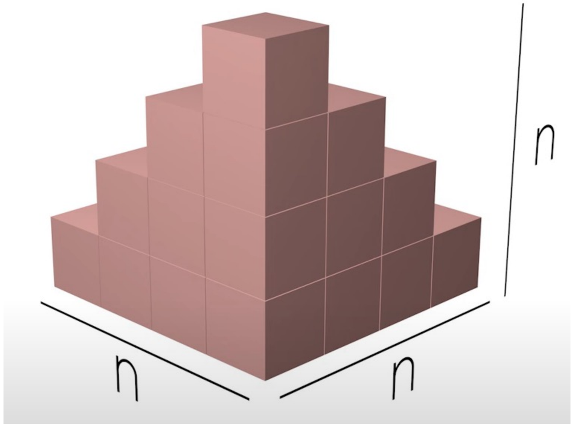
T = 2m

p = 2

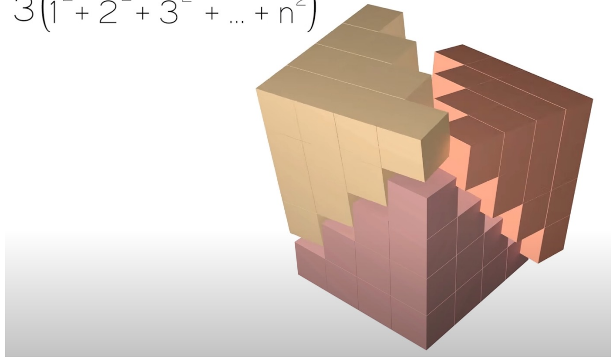
I will represent each term in the series with equivalent number of cubes



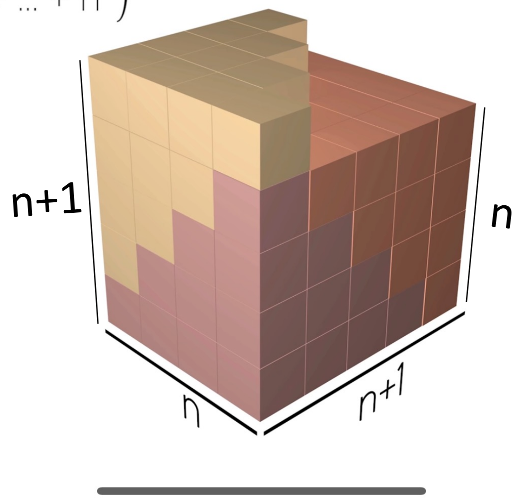
$1^2 + 2^2 + 3^2 + \ldots + n^2$



$3(1^2 + 2^2 + 3^2 + \ldots + n^2)$



$3(1^2 + 2^2 + 3^2 + \ldots + n^2)$



length = n

width = n + 1

$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots + n^2$

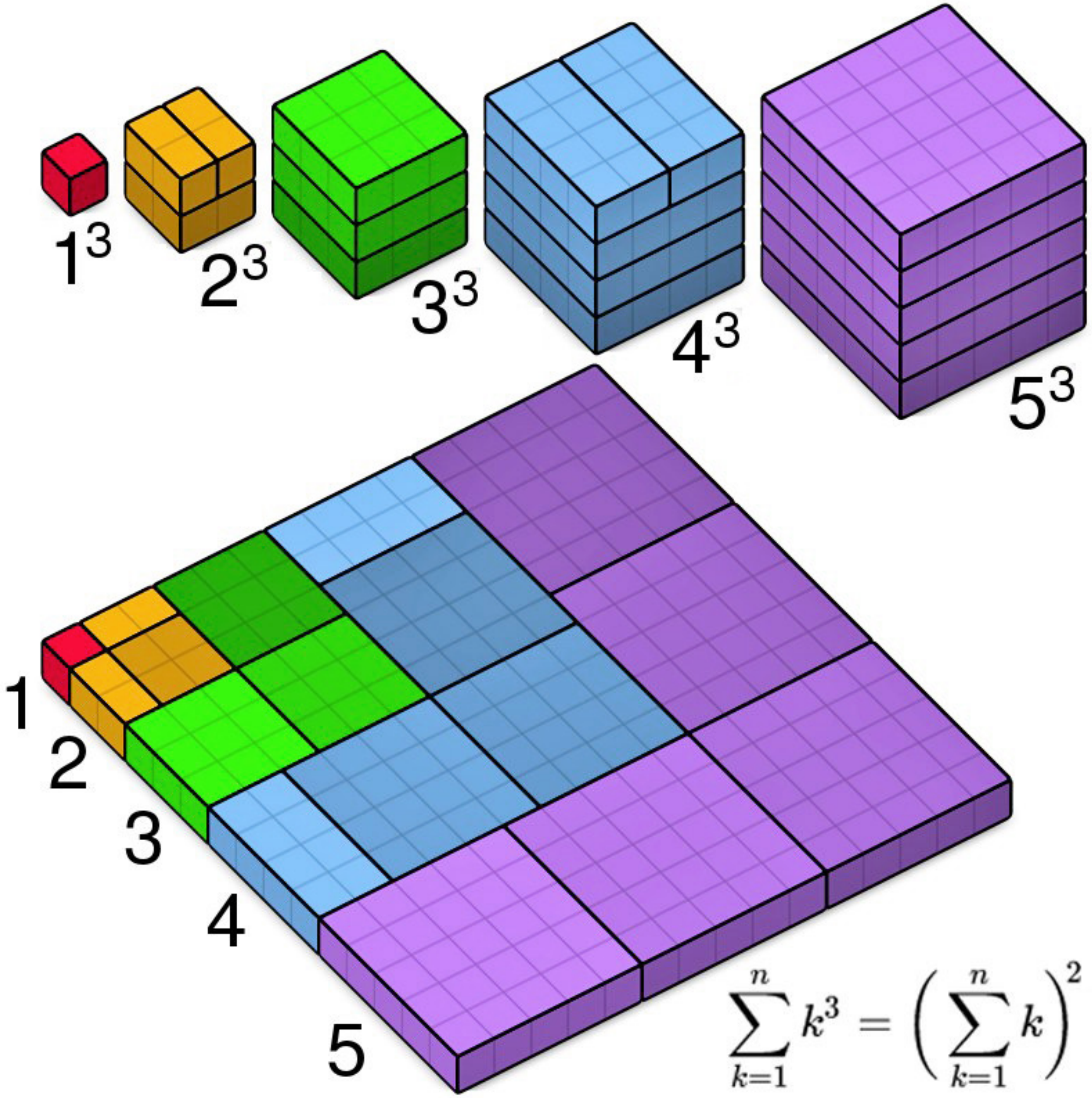
is equal to

$$\frac{(n(n+1)(2n+1))}{6} = S_2(n) = \tfrac{1}{3} n^3 + \tfrac{1}{2} n^2 + \tfrac{1}{6} n$$

https://www.youtube.com/watch?v=aXbT37IlyZQ

T = 2m30s

p = 3



$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$$

$= S_3(n)$

$= (S_1(n))^2 = \frac{1}{4} n^4 + \frac{1}{2} n^3 + \frac{1}{4} n^2$

$+ 0n$

T = 2m50s

p = 4

In order to find the formula for S_4 we begin with the identity $(a+1)^5 = a^5 + 5a^4 + 10a^3 + 10a^2 + 5a + 1$ to write the following equations

$$\begin{array}{rcllllll}
(n+1)^5 &=& n^5 & +5n^4 & +10n^3 & +10n^2 & +5n & +1 \\
n^5 &=& (n-1)^5 & +5(n-1)^4 & +10(n-1)^3 & +10(n-1)^2 & +5(n-1) & +1 \\
(n-1)^5 &=& (n-2)^5 & +5(n-2)^4 & +10(n-2)^3 & +10(n-2)^2 & +5(n-2) & +1 \\
& \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
4^5 &=& 3^5 & +5\cdot 3^4 & +10\cdot 3^3 & +10\cdot 3^2 & +5\cdot 3 & +1 \\
3^5 &=& 2^5 & +5\cdot 2^4 & +10\cdot 2^3 & +10\cdot 2^2 & +5\cdot 2 & +1 \\
2^5 &=& 1^5 & +5\cdot 1^4 & +10\cdot 1^3 & +10\cdot 1^2 & +5\cdot 1 & +1
\end{array}$$

After adding these n equations and canceling the numbers $2^5, 3^5, \ldots, n^5$ that show up to the left and to the right of "=" sign, the resulting equation becomes

$$(n+1)^5 = 1^5 + 5\sum_{k=1}^{n} k^4 + 10\sum_{k=1}^{n} k^3 + 10\sum_{k=1}^{n} k^2 + 5\sum_{k=1}^{n} k + 1\cdot n$$

or

$$(n+1)^5 = 1 + 5S_4 + 10S_3 + 10S_2 + 5S_1 + n.$$

Using that $S_1 = \frac{n(n+1)}{2}$, $S_2 = \frac{n(n+1)(2n+1)}{6}$, and $S_3 = \left[\frac{n(n+1)}{2}\right]^2$ to solve for S_4 results in

$$S_4 = \frac{(n+1)^5 - 10\left[\frac{n(n+1)}{2}\right]^2 - 10\frac{n(n+1)(2n+1)}{6} - 5\frac{n(n+1)}{2} - n - 1}{5}$$

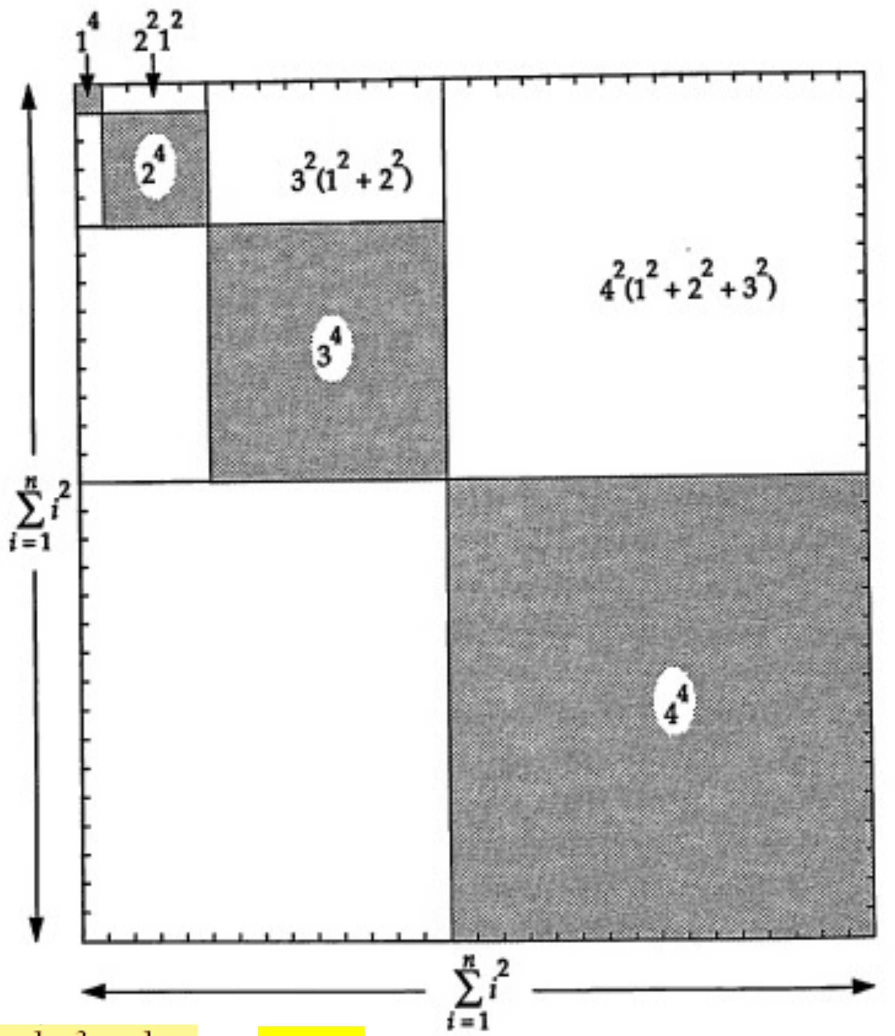
which can be simplified to

$$S_4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$ $= (S_4(n) = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$

-1/30

Sums of Fourth Powers

$$\sum_{i=1}^{n} i^4 = \left(\sum_{i=1}^{n} i^2\right)^2 - 2\left[\sum_{k=2}^{n}\left(k^2 \sum_{i=1}^{k-1} i^2\right)\right]$$



T = 3m30s

—Elizabeth M. Markham

$$A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 5 & 10 & 10 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 6 & 15 & 20 & 15 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 11 & 55 & 165 & 330 & 462 & 462 & 330 & 165 & 55 & 11 & 0 & 0 & 0 & 0 & 0 \\
1 & 12 & 66 & 220 & 495 & 792 & 924 & 792 & 495 & 220 & 66 & 12 & 0 & 0 & 0 & 0 \\
1 & 13 & 78 & 286 & 715 & 1287 & 1716 & 1716 & 1287 & 715 & 286 & 78 & 13 & 0 & 0 & 0 \\
1 & 14 & 91 & 364 & 1001 & 2002 & 3003 & 3432 & 3003 & 2002 & 1001 & 364 & 91 & 14 & 0 & 0 \\
1 & 15 & 105 & 455 & 1365 & 3003 & 5005 & 6435 & 6435 & 5005 & 3003 & 1365 & 455 & 105 & 15 & 0 \\
1 & 16 & 120 & 560 & 1820 & 4368 & 8008 & 11440 & 12870 & 11440 & 8008 & 4368 & 1820 & 560 & 120 & 16
\end{bmatrix}$$

$e := 16 \quad n := 0..e \quad k := 0..e-1$

$$A_{n,k} := \begin{cases} \text{if } n > k & \dfrac{(n)!}{k! \cdot (n-k)!} \\ \text{else} & 0 \end{cases} \qquad A := \text{submatrix}(A, 1, e, 0, e-1)$$

T = 4m

0	1
1	$\pm\frac{1}{2}$
2	$\frac{1}{6}$
3	0
4	$-\frac{1}{30}$
5	0
6	$\frac{1}{42}$
7	0
8	$-\frac{1}{30}$
9	0
10	$\frac{5}{66}$
11	0
12	$-\frac{691}{2730}$
13	0
14	$\frac{7}{6}$
15	0

$$A^{-1} \rightarrow \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & -\frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{30} & 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{12} & 0 & \frac{5}{12} & -\frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{42} & 0 & -\frac{1}{6} & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{12} & 0 & -\frac{7}{24} & 0 & \frac{7}{12} & -\frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{30} & 0 & \frac{2}{9} & 0 & -\frac{7}{15} & 0 & \frac{2}{3} & -\frac{1}{2} & \frac{1}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{3}{20} & 0 & \frac{1}{2} & 0 & -\frac{7}{10} & 0 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{10} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{5}{66} & 0 & -\frac{1}{2} & 0 & 1 & 0 & -1 & 0 & \frac{5}{6} & -\frac{1}{2} & \frac{1}{11} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{5}{12} & 0 & -\frac{11}{8} & 0 & \frac{11}{6} & 0 & -\frac{11}{8} & 0 & \frac{11}{12} & -\frac{1}{2} & \frac{1}{12} & 0 & 0 & 0 & 0 \\
-\frac{691}{2730} & 0 & \frac{5}{3} & 0 & -\frac{33}{10} & 0 & \frac{22}{7} & 0 & -\frac{11}{6} & 0 & 1 & -\frac{1}{2} & \frac{1}{13} & 0 & 0 & 0 \\
0 & -\frac{691}{420} & 0 & \frac{65}{12} & 0 & -\frac{143}{20} & 0 & \frac{143}{28} & 0 & -\frac{143}{60} & 0 & \frac{13}{12} & -\frac{1}{2} & \frac{1}{14} & 0 & 0 \\
\frac{7}{6} & 0 & -\frac{691}{90} & 0 & \frac{91}{6} & 0 & -\frac{143}{10} & 0 & \frac{143}{18} & 0 & -\frac{91}{30} & 0 & \frac{7}{6} & -\frac{1}{2} & \frac{1}{15} & 0 \\
0 & \frac{35}{4} & 0 & -\frac{691}{24} & 0 & \frac{455}{12} & 0 & -\frac{429}{16} & 0 & \frac{143}{12} & 0 & -\frac{91}{24} & 0 & \frac{5}{4} & -\frac{1}{2} & \frac{1}{16}
\end{pmatrix}$$

T = 4m20s

jupyterlite **Bernoulli Numbers**

File Edit View Run Kernel Settings Help

Code

[1]:

```python
from fractions import Fraction
import time
s = time.time()
N=100

def Bernoulli():
    A, m = [], 0
    while True:
        A.append(Fraction(1, m+1))
        for j in range(m, 0, -1):
            A[j-1] = j*(A[j-1] - A[j])
        yield A[0]
        m += 1

Bn = [ix for ix in zip(range(N+1), Bernoulli())]
Bn = [(i, b) for i,b in Bn if b]

width = max(len(str(b.numerator)) for i,b in Bn)
for i,b in Bn:
    print('B(%2i) = %*i/%i' % (i, width, b.numerator, b.denominator))
e = time.time() - s
print()
print ('Time is %f seconds' % e)
```

```
B( 0) = 1/1
B( 1) = 1/2
B( 2) = 1/6
B( 4) = -1/30
B( 6) = 1/42
B( 8) = -1/30
B(10) = 5/66
B(12) = -691/2730
B(14) = 7/6
B(16) = -3617/510
B(18) = 43867/798
B(20) = -174611/330
B(22) = 854513/138
B(24) = -236364091/2730
B(26) = 8553103/6
B(28) = -23749461029/870
B(30) = 8615841276005/14322
B(32) = -7709321041217/510
B(34) = 2577687858367/6
B(36) = -26315271553053477373/1919190
B(38) = 2929993913841559/6
B(40) = -261082718496449122051/13530
B(42) = 1520097643918070802691/1806
B(44) = -27833269579301024235023/690
B(46) = 596451111593912163277961/282
B(48) = -5609403368997817686249127547/46410
B(50) = 495057205241079648212477525/66
B(52) = -801165718135489957347924991853/1590
B(54) = 29149963634884862421418123812691/798
B(56) = -2479392929313226753685415739663229/870
B(58) = 84483613348880041862046775994036021/354
B(60) = -1215233140483755572040304994079820246041491/56786730
B(62) = 12300585434086858541953039857403386151/6
B(64) = -106783830147866529886385444979142647942017/510
B(66) = 1472600022126335654051619428551932342241899101/64722
B(68) = -78773130858718728141909149208474606244347001/30
B(70) = 1505381347333367003803076567377857208511438160235/4686
B(72) = -5827954961669944110438277244641067365282488301844260429/140100870
B(74) = 34152417289221168014330073731472635186688307783087/6
B(76) = -24655088825935372707687196040585199904365267828865801/30
B(78) = 414846365575400828295179035549542073492199375372400483487/3318
B(80) = -4603784299479457646935574969019046849794257872751288919656867/230010
B(82) = 1677014149185145836823154509786269900207736027570253414881613/498
B(84) = -2024576195935290360231131160111731009989917391198090877281083932477/3404310
B(86) = 660714619417678653573847847426261496277830686653388931761996983/6
B(88) = -1311426488674017507995511424019311843345750275572028644296919890574047/61410
B(90) = 1179057279021082799884123351249215083775254949669647116231545215727922535/272118
B(92) = -1295585948207537527989427828538576749659341483719435143023316326829946247/1410
B(94) = 1220813806579744469607301679413201203958508415202696621436215105284649447/6
B(96) = -211600449597266513097597728109824233673043954389060234150638733420050668349987259/4501770
B(98) = 67908260672905495624051117546403605607342195728504487509073961249992947058239/6
B(100) = -94598037819122125295227433069493721872702841533066936133385696204311395415197247711/33330

Time is 0.054000 seconds
```

https://rosettacode.org/wiki/Bernoulli_numbers

T = 4m40s

Bug or Typesetting error ?

Should read V_4 / V_5

Diagram for the computation by the Engine of the Numbers of Bernoulli. See Note G. (page 722 *et seq.*)

Number of Operation.	Nature of Operation.	Variables acted upon.	Variables receiving results.	Indication of change in the value on any Variable.	Statement of Results.	Data. ${}^1V_1$ 0 0 0 1 [1]	${}^1V_2$ 0 0 0 2 [2]	${}^1V_3$ 0 0 0 4 [n]	Working Variables. ${}^0V_4$ 0 0 0 0	${}^0V_5$ 0 0 0 0	${}^0V_6$ 0 0 0 0	${}^0V_7$ 0 0 0 0	${}^0V_8$ 0 0 0 0	${}^0V_9$ 0 0 0 0	${}^0V_{10}$ 0 0 0 0	${}^0V_{11}$ 0 0 0 0	${}^0V_{12}$ 0 0 0 0	${}^0V_{13}$ 0 0 0 0	Result Variables. ${}^1V_{21}$ B_1 in a decimal fraction. [B_1]	${}^1V_{22}$ B_3 in a decimal fraction. [B_3]	${}^1V_{23}$ B_5 in a decimal fraction. [B_5]	${}^0V_{24}$ 0 0 0 0 [B_7]
1	×	${}^1V_2 \times {}^1V_3$	${}^1V_4, {}^1V_5, {}^1V_6$	$\left\{\begin{matrix}{}^1V_2 = {}^1V_2 \\ {}^1V_3 = {}^1V_3\end{matrix}\right\}$	$= 2n$	...	2	n	$2n$	$2n$	$2n$											
2	−	${}^1V_4 - {}^1V_1$	${}^2V_4$	$\left\{\begin{matrix}{}^1V_4 = {}^2V_4 \\ {}^1V_1 = {}^1V_1\end{matrix}\right\}$	$= 2n - 1$	1	...	...	$2n-1$													
3	+	${}^1V_5 + {}^1V_1$	${}^2V_5$	$\left\{\begin{matrix}{}^1V_5 = {}^2V_5 \\ {}^1V_1 = {}^1V_1\end{matrix}\right\}$	$= 2n + 1$	1	...	...	...	$2n+1$												
4	÷	${}^2V_5 \div {}^2V_4$	${}^1V_{11}$	$\left\{\begin{matrix}{}^2V_5 = {}^0V_5 \\ {}^2V_4 = {}^0V_4\end{matrix}\right\}$	$= \frac{2n-1}{2n+1}$	...	...	...	0	0	...	...	...	...	...	$\frac{2n-1}{2n+1}$						
5	÷	${}^1V_{11} \div {}^1V_2$	${}^2V_{11}$	$\left\{\begin{matrix}{}^1V_{11} = {}^2V_{11} \\ {}^1V_2 = {}^1V_2\end{matrix}\right\}$	$= \frac{1}{2} \cdot \frac{2n-1}{2n+1}$	...	2	...	...	...	...	...	...	...	...	$\frac{1}{2} \cdot \frac{2n-1}{2n+1}$						
6	−	${}^0V_{13} - {}^2V_{11}$	${}^1V_{13}$	$\left\{\begin{matrix}{}^2V_{11} = {}^0V_{11} \\ {}^0V_{13} = {}^1V_{13}\end{matrix}\right\}$	$= -\frac{1}{2} \cdot \frac{2n-1}{2n+1} = A_0$	...	...	...	...	...	...	...	...	...	...	0	....	$-\frac{1}{2} \cdot \frac{2n-1}{2n+1} = A_0$				
7	−	${}^1V_3 - {}^1V_1$	${}^1V_{10}$	$\left\{\begin{matrix}{}^1V_3 = {}^1V_3 \\ {}^1V_1 = {}^1V_1\end{matrix}\right\}$	$= n - 1 \ (= 3)$	1	...	n	...	...	...	...	...	...	$n-1$							
8	+	${}^1V_2 + {}^0V_7$	${}^1V_7$	$\left\{\begin{matrix}{}^1V_2 = {}^1V_2 \\ {}^0V_7 = {}^1V_7\end{matrix}\right\}$	$= 2 + 0 = 2$	...	2	...	...	...	...	2										
9	÷	${}^1V_6 \div {}^1V_7$	${}^3V_{11}$	$\left\{\begin{matrix}{}^1V_6 = {}^1V_6 \\ {}^0V_{11} = {}^3V_{11}\end{matrix}\right\}$	$= \frac{2n}{2} = A_1$	...	...	...	...	...	$2n$	2	...	...	...	$\frac{2n}{2} = A_1$						
10	×	${}^1V_{21} \times {}^3V_{11}$	${}^1V_{12}$	$\left\{\begin{matrix}{}^1V_{21} = {}^1V_{21} \\ {}^3V_{11} = {}^3V_{11}\end{matrix}\right\}$	$= B_1 \cdot \frac{2n}{2} = B_1 A_1$	...	...	...	...	...	...	...	...	...	...	$\frac{2n}{2} = A_1$	$B_1 \cdot \frac{2n}{2} = B_1 A_1$	....	B_1			
11	+	${}^1V_{12} + {}^1V_{13}$	${}^2V_{13}$	$\left\{\begin{matrix}{}^1V_{12} = {}^0V_{12} \\ {}^1V_{13} = {}^2V_{13}\end{matrix}\right\}$	$= -\frac{1}{2} \cdot \frac{2n-1}{2n+1} + B_1 \cdot \frac{2n}{2}$ ...	...	...	...	...	...	...	...	...	...	...	....	0	$\left\{-\frac{1}{2} \cdot \frac{2n-1}{2n+1} + B_1 \cdot \frac{2n}{2}\right\}$				
12	−	${}^1V_{10} - {}^1V_1$	${}^2V_{10}$	$\left\{\begin{matrix}{}^1V_{10} = {}^2V_{10} \\ {}^1V_1 = {}^1V_1\end{matrix}\right\}$	$= n - 2 \ (= 2)$	1	...	...	...	...	...	...	...	...	$n-2$							
13	−	${}^1V_6 - {}^1V_1$	${}^2V_6$	$\left\{\begin{matrix}{}^1V_6 = {}^2V_6 \\ {}^1V_1 = {}^1V_1\end{matrix}\right\}$	$= 2n - 1$	1	...	...	...	...	$2n-1$											
14	+	${}^1V_1 + {}^1V_7$	${}^2V_7$	$\left\{\begin{matrix}{}^1V_1 = {}^1V_1 \\ {}^1V_7 = {}^2V_7\end{matrix}\right\}$	$= 2 + 1 = 3$	1	...	...	...	...	...	3										
15	÷	${}^2V_6 \div {}^2V_7$	${}^1V_8$	$\left\{\begin{matrix}{}^2V_6 = {}^2V_6 \\ {}^2V_7 = {}^2V_7\end{matrix}\right\}$	$= \frac{2n-1}{3}$	...	...	...	...	...	$2n-1$	3	$\frac{2n-1}{3}$									
16	×	${}^1V_8 \times {}^3V_{11}$	${}^4V_{11}$	$\left\{\begin{matrix}{}^1V_8 = {}^0V_8 \\ {}^3V_{11} = {}^4V_{11}\end{matrix}\right\}$	$= \frac{2n}{2} \cdot \frac{2n-1}{3}$	...	...	...	...	...	...	...	0	...	...	$\frac{2n}{2} \cdot \frac{2n-1}{3}$						
17	−	${}^2V_6 - {}^1V_1$	${}^3V_6$	$\left\{\begin{matrix}{}^2V_6 = {}^3V_6 \\ {}^1V_1 = {}^1V_1\end{matrix}\right\}$	$= 2n - 2$	1	...	...	...	...	$2n-2$											
18	+	${}^1V_1 + {}^2V_7$	${}^3V_7$	$\left\{\begin{matrix}{}^2V_7 = {}^3V_7 \\ {}^1V_1 = {}^1V_1\end{matrix}\right\}$	$= 3 + 1 = 4$	1	...	...	...	...	...	4										
19	÷	${}^3V_6 \div {}^3V_7$	${}^1V_9$	$\left\{\begin{matrix}{}^3V_6 = {}^3V_6 \\ {}^3V_7 = {}^3V_7\end{matrix}\right\}$	$= \frac{2n-2}{4}$	...	...	...	...	...	$2n-2$	4	...	$\frac{2n-2}{4}$	...	$\left\{\frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{3} = A_3\right\}$						
20	×	${}^1V_9 \times {}^4V_{11}$	${}^5V_{11}$	$\left\{\begin{matrix}{}^1V_9 = {}^0V_9 \\ {}^4V_{11} = {}^5V_{11}\end{matrix}\right\}$	$= \frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{4} = A_3$	...	...	...	...	...	...	...	...	0								
21	×	${}^1V_{22} \times {}^5V_{11}$	${}^0V_{12}$	$\left\{\begin{matrix}{}^1V_{22} = {}^1V_{22} \\ {}^0V_{12} = {}^2V_{12}\end{matrix}\right\}$	$= B_3 \cdot \frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{3} = B_3 A_3$	...	...	...	...	...	...	...	...	...	...	0	$B_3 A_3$	....	......	B_3		
22	+	${}^2V_{12} + {}^2V_{13}$	${}^3V_{13}$	$\left\{\begin{matrix}{}^2V_{12} = {}^0V_{12} \\ {}^2V_{13} = {}^3V_{13}\end{matrix}\right\}$	$= A_0 + B_1 A_1 + B_3 A_3$	...	...	...	...	...	...	...	...	...	...	....	0	$\left\{A_3 + B_1 A_1 + B_3 A_3\right\}$				
23	−	${}^2V_{10} - {}^1V_1$	${}^3V_{10}$	$\left\{\begin{matrix}{}^2V_{10} = {}^3V_{10} \\ {}^1V_1 = {}^1V_1\end{matrix}\right\}$	$= n - 3 \ (= 1)$	1	...	...	...	...	...	...	...	...	$n-3$							
					Here follows a repetition of Operations thirteen to twenty-three.																	
24	+	${}^4V_{13} + {}^0V_{24}$	${}^1V_{24}$	$\left\{\begin{matrix}{}^4V_{13} = {}^0V_{13} \\ {}^0V_{24} = {}^1V_{24}\end{matrix}\right\}$	$= B_7$	...	...	...	...	...	...	...	...	...	...	....	....	....	......	......	......	B_7
25	+	${}^1V_1 + {}^1V_3$	${}^1V_3$	$\left\{\begin{matrix}{}^1V_1 = {}^1V_1 \\ {}^1V_3 = {}^1V_3 \\ {}^5V_6 = {}^0V_6 \\ {}^5V_7 = {}^0V_7\end{matrix}\right\}$	$= n + 1 = 4 + 1 = 5$ .... by a Variable-card. by a Variable card.	1	...	$n+1$	...	...	0	0										

Ada Lovelace Note G
Program to find 'B_7' (- 1/30)

https://enigmaticcode.wordpress.com/tag/bernoulli-numbers/
https://twobithistory.org/2018/08/18/ada-lovelace-note-g.html
https://writings.stephenwolfram.com/2015/12/untangling-the-tale-of-ada-lovelace/

T = 5m

Further Reading

The Bernoulli Numbers: A Brief Primer

Nathaniel Larson

May 10, 2019

On the sums of series of reciprocals*

Leonhard Euler

SUM OF POWERS OF THE FIRST N INTEGERS

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Obiamaka Agu

December 2020

https://www.whitman.edu/documents/Academics/Mathematics/2019/Larson-Balof.pdf
https://scholarworks.lib.csusb.edu/etd/1132/
https://arxiv.org/abs/math/0506415

… and what if p is not an integer or negative
(i.e. p = - ½ - the Basel problem, Euler later proved $S_p(n)$ tends as n goes to infinity to $\pi^2/6$) ?
and connections with to the Riemann zeta function

references