

# Abundant Number Theory

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# What are abundant numbers?

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- E.g.  $s(10) = 1 + 2 + 5 = 8$ ,  $s(11) = 1$ ,  $s(12) = 1 + 2 + 3 + 4 + 6 = 16$ .

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## Definition

An abundant number  $n$  is a natural number with  $s(n) > n$ .

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## Proof.

Let the proper factors of  $a$  be  $a_1, \dots, a_k$ . We know  $a_1 + \dots + a_k > a$ . Let  $ma$  be a multiple of  $a$ . For any  $a_i$ ,  $ma_i$  is a factor of  $ma$ . So

$$s(ma) \geq ma_1 + \dots + ma_k = ms(a) > ma. \quad \square$$

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# Primitive Abundant Numbers

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- The first primitive abundant numbers are 12, 18, 20, 30, 42.
- There are infinitely many primitive abundant numbers<sup>2</sup>.

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- Let  $a(n)$  be the number of abundant numbers  $\leq n$ .
- What happens to  $a(n)/n$  as  $n \rightarrow \infty$ ?

Let  $A$  be the limit of  $a(n)/n$ :

$$A = \lim_{n \rightarrow \infty} \frac{a(n)}{n}.$$

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- ①  $0 \leq A \leq 0.5$ ,
- ②  $0.5 < A \leq 1$ ,

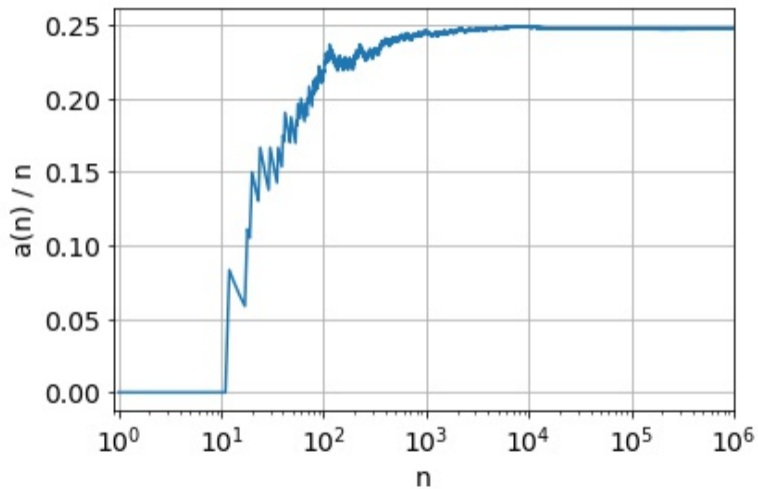
Let  $A$  be the limit of  $a(n)/n$ :

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Which of these three statements is true?

- 1  $0 \leq A \leq 0.5$ ,
- 2  $0.5 < A \leq 1$ ,
- 3 The limit does not exist!

# Natural Density of the Abundant Numbers



# Convergence of the Limit

- Davenport proved that the limit exists in 1933.
- Deléglise proved that  $0.2474 < A < 0.2480$  in 1998.

# Calculating the Limit

- Think about the sequence of primitive abundant numbers.
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# Calculating the Limit

- Think about the sequence of primitive abundant numbers.
- Every multiple of 12 is abundant so  $A \geq 1/12$ .
- Every multiple of 18 is abundant so  $A \geq 1/12 + 1/18 - 1/(12 \cdot 18)$ .
- The limit of this sequence is  $A$ , and the sequence is increasing slowly.

- Any multiple of an abundant number is itself abundant.

# Main Results

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- There are infinitely many odd and even abundant numbers.

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- Any multiple of an abundant number is itself abundant.
- There are infinitely many odd and even abundant numbers.
- Abundant numbers make up about a quarter of the natural numbers.