The incomprehensible integer TREE(3)

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Summary

- I’ll tell you a little about the number TREE(3)
- I found out about it in Antonio Padilla, *Fantastic Numbers: A Journey to the Edge of Physics*
- I was amazed that nobody had told me about it before
- Apologies if you know all this already!

The Game of Trees

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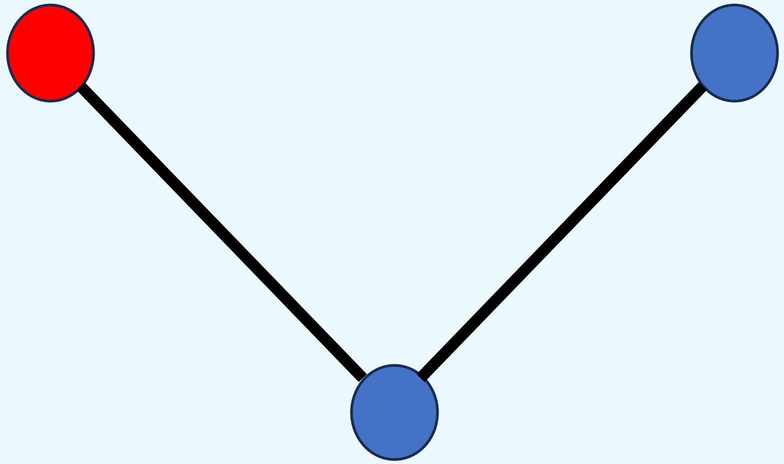
The Game of Trees

- You have a finite choice of colours (say red, blue and green)
- You construct a forest: on each turn you add a new tree by connecting nodes of these colours with lines



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Rules

- On move one your tree must have at most one node
- On move 2 it must have at most 2 nodes
- On move *n* it must have at most *n* nodes

- If a new tree in the forest *contains* a tree constructed earlier, game is over
- *Contains* has a natural definition which I will skip

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The Game of Trees with one colour or two colours

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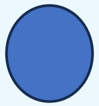
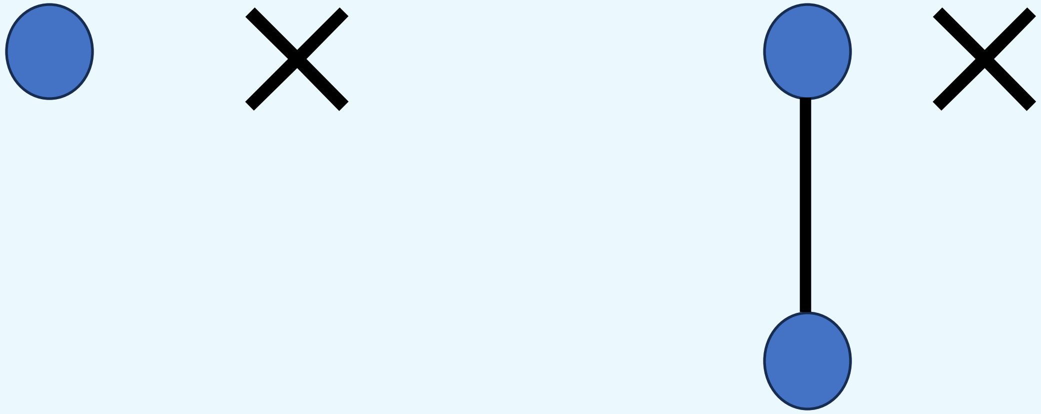
One Colour

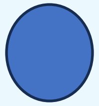
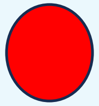
- Move 1:



- Move 2:



Two Colours

- Move 1: 
- Move 2: 
- Move 3: 
- Move 4: ?

Must this game end?

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A finite game:

- Conjecture (Andrew Vázsonyi):
  for any finite number of colours,
  this game must end in a finite
  number of moves
- Proved by Joseph Kruskal (1960)

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How many moves can it last?

- Let TREE(*n*) be the maximum number of moves that the game with *n* colours can last

- TREE(1) = 1
- TREE(2) = 3
- But we don’t know TREE(3)

But we do have a lower bound for TREE(3), thanks to Harvey Friedman

Knuth's Arrow Notation

- $a \uparrow b = \ a^b$
- $a \uparrow\uparrow b = \ a \uparrow (a \uparrow (\dots \uparrow a))$

  (with *b* repetitions)
- $a \uparrow\uparrow\uparrow b = \ a \uparrow\uparrow (a \uparrow\uparrow (\dots \uparrow\uparrow a))$

  (with *b* repetitions)

- Examples:
- 3↑3 = 27
- 3↑↑3 = 7,625,597,484,987
- 3↑↑4 = $3^{7,625,597,484,987}$
- 3↑↑↑3 = 3 ↑↑ 7,625,597,484,987

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Graham’s Number

- $g_1 = 3 \uparrow\uparrow\uparrow\uparrow 3$
- $g_2 = 3 \uparrow^{g_1} 3$ (g_1 arrows)
- $g_3 = 3 \uparrow^{g_2} 3$ (g_2 arrows)

- Graham’s Number is g_{64}

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Lower bond for TREE(3)

- 64 steps up the chain takes you to Graham's number
- $2 \uparrow\uparrow\uparrow \cdots \uparrow 187{,}196$ (187,195 arrows) **steps** up the chain takes you to the lower bound for TREE(3)

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Counclusion: TREE(3) is an incomprehensively large number!

But 100% of positive integers are larger than TREE(3)

So TREE(3) is an unusually small number

Sources

- Antonio Padilla, *Fantastic Numbers and where to find them: A Journey to the Edge of Physics* (Penguiin 2023)

- Numberphile: The Enormous TREE(3) - https://www.youtube.com/watch?v=3P6DWAwwViU
- and TREE(3) Extra Footage – https://www.youtube.com/watch?v=IihcNa9YAPk

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Thank you

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