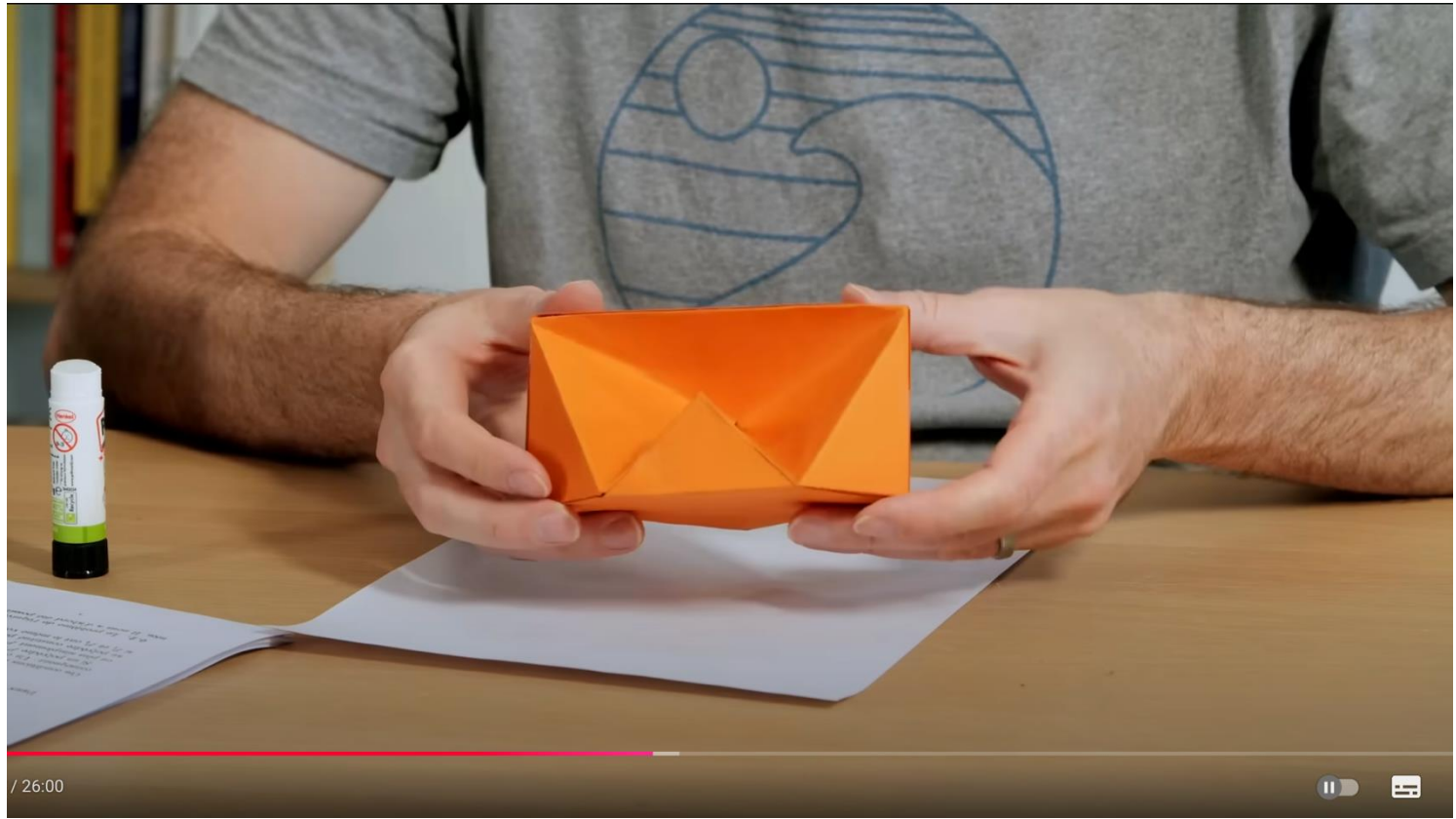


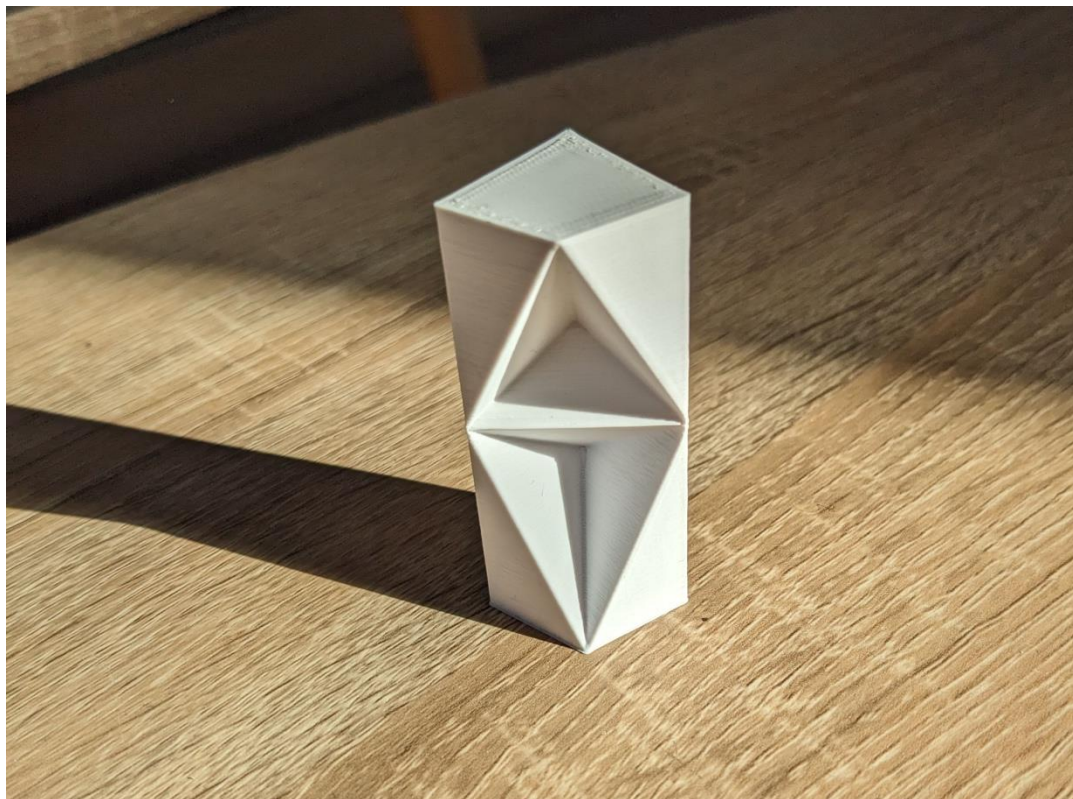


The shape that should be impossible.

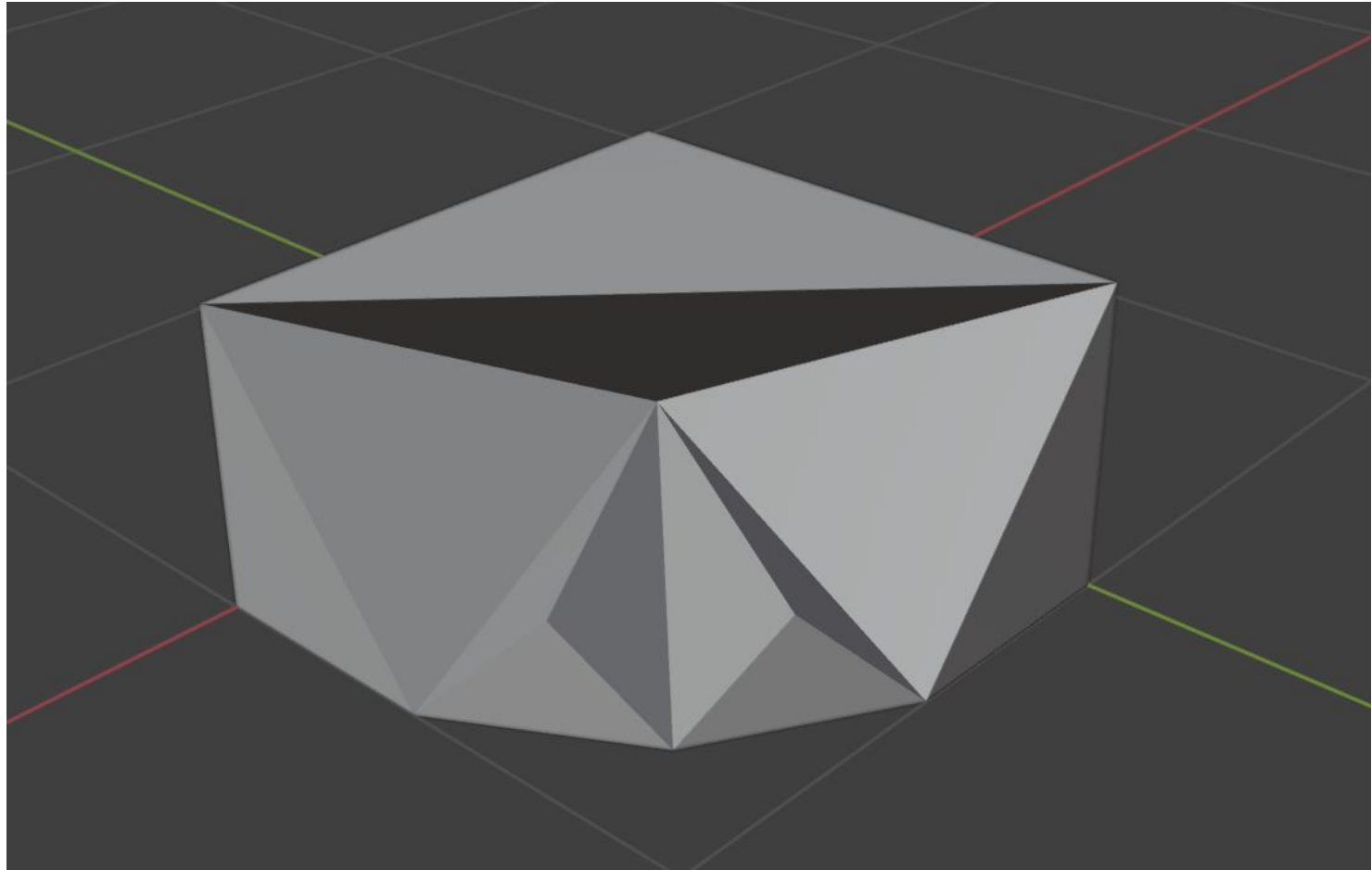
Stand-up Maths, 5 Jun 2024





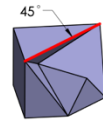


James Cordery



Daniel Bernhardsson
Cow mouth shapes

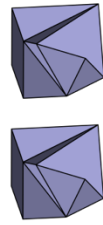
Shape Construction



Step-1

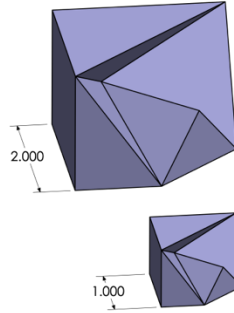
Starting shape

Robin's Shape in this example, but can be any later extension of this construction



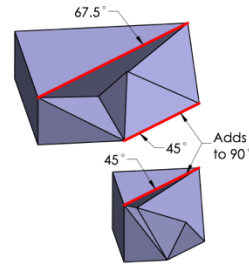
Step-2

Add Robin's Shape above the starting shape.



Step-3

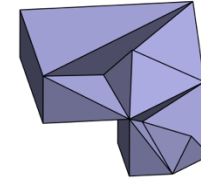
Scale the top shape so that the base is double that of the starting shape.



Step-4

Adjust the front edge angle to complement the reduced angle of the starting shape while maintaining all other right angles.

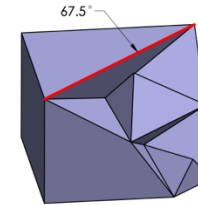
This has the effect of increasing the 45° angle of Robin's Shape.



Step-5

Merge the shapes such that the complementary angles form a right angle.

This shape now meets the criteria of a single reduced angle, however we will perform 1 additional step to simplify the construction.



Step-6

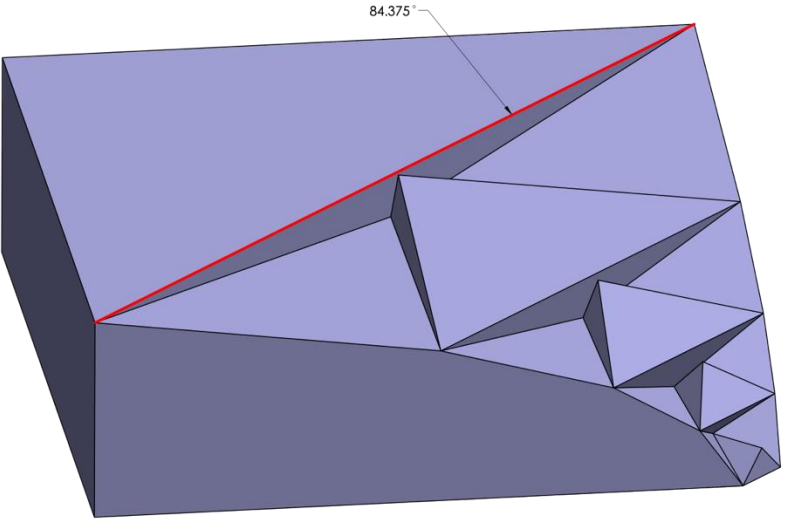
Extend the surface down to remove unnecessary geometry and simplify the construction.

Brian Williams

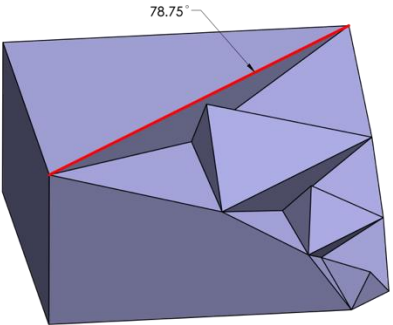
Construction Method for Infinite *Impossible* Shapes

Shape Extensions

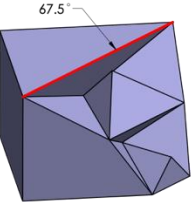
	Angle
Robin's Shape	45.0000000000
Extension 1	67.5000000000
Extension 2	78.7500000000
Extension 3	84.3750000000
Extension 4	87.1875000000
Extension 5	88.5937500000
Extension 6	89.2968750000
Extension 7	89.6484375000
Extension 8	89.8242187500
Extension 9	89.9121093750
Extension 10	89.9560546875
Extension n	$\theta_n = \theta_{n-1} + 45(2^{-n})$



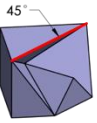
Extension 3



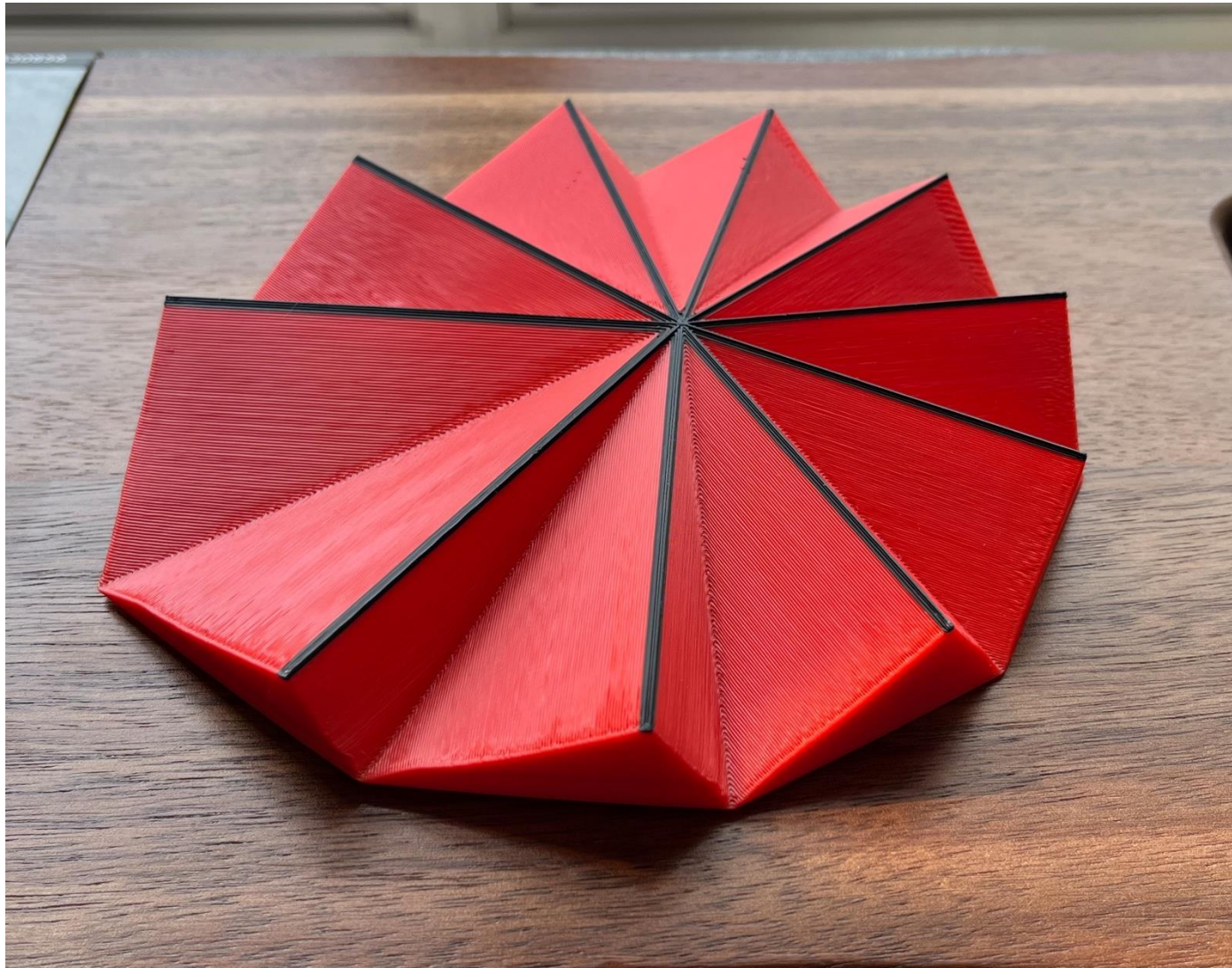
Extension 2



Extension 1



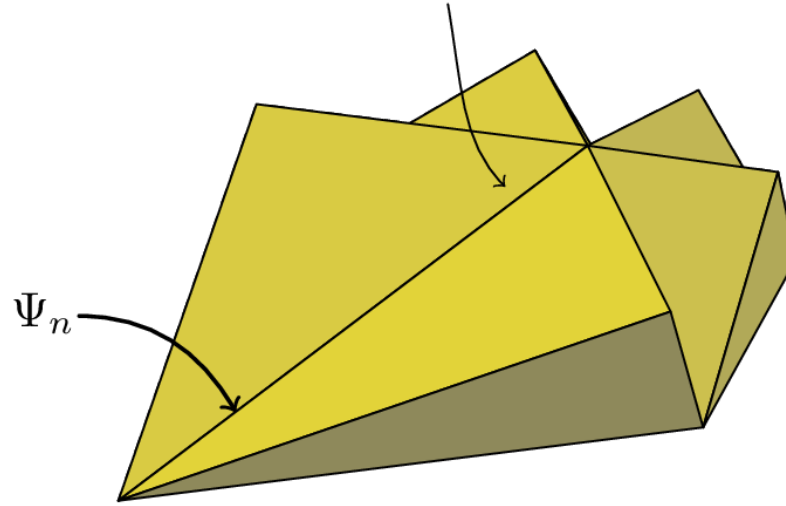
Robin's Shape



Michael Engen
@mengen@mastodon.social

Unique-valley Shape

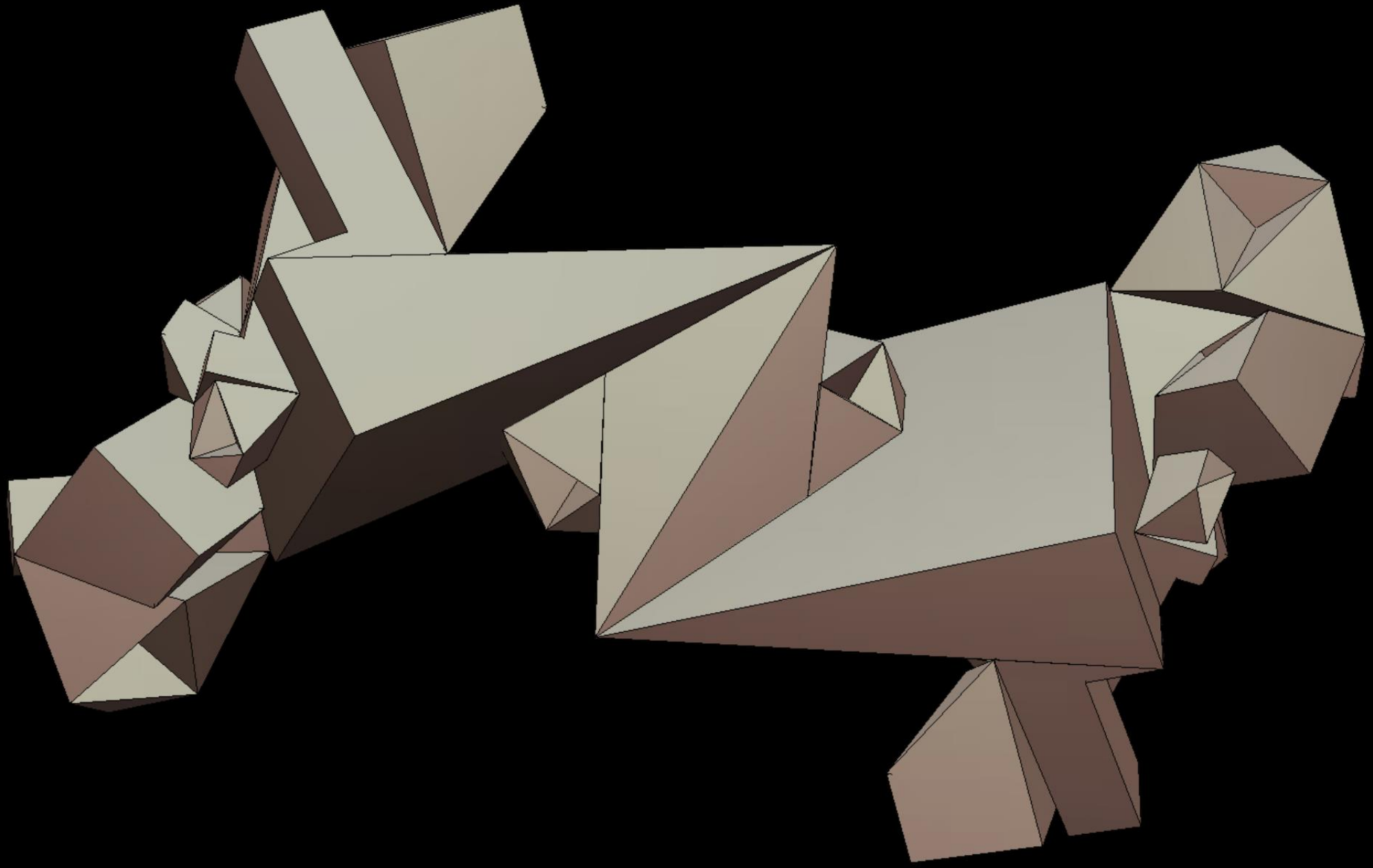
$n \geq 4$ valleys



$$\cos \Psi_n = \frac{\cos^{2n} \psi_n - 1}{\cos^{2n} \psi_n + 2 \cos^{n+1} \psi_n + 1}$$

where $\psi_n < 2\pi/n$ satisfies

$$\frac{\cos(n-1)\psi_n - 1}{\sin(n-1)\psi_n} = \frac{\sin \psi_n}{\cos \psi_n} \frac{1}{\cos^{n-1} \psi_n + 1}$$



FIRST SHAPE DISCOVERED BY COMPUTER



The search for the biggest shape in the universe.



Stand-up Maths

1.23M subscribers



@andriypredmyrskyy7791 1 month ago

YouTube comment by **Andriy Predmyrskyy**

I'm a little upset we didn't get to hear more details about the shape, I barely even know that it looks like! (Pink, presumably)

👍 170 💬 Reply

[This short talk will tell you more about what it looks like 😊](#)

A search for *the biggest* shape in *the universe*

where:

the universe has three spatial dimensions

the universe is spherical

the radius of the universe is **1**

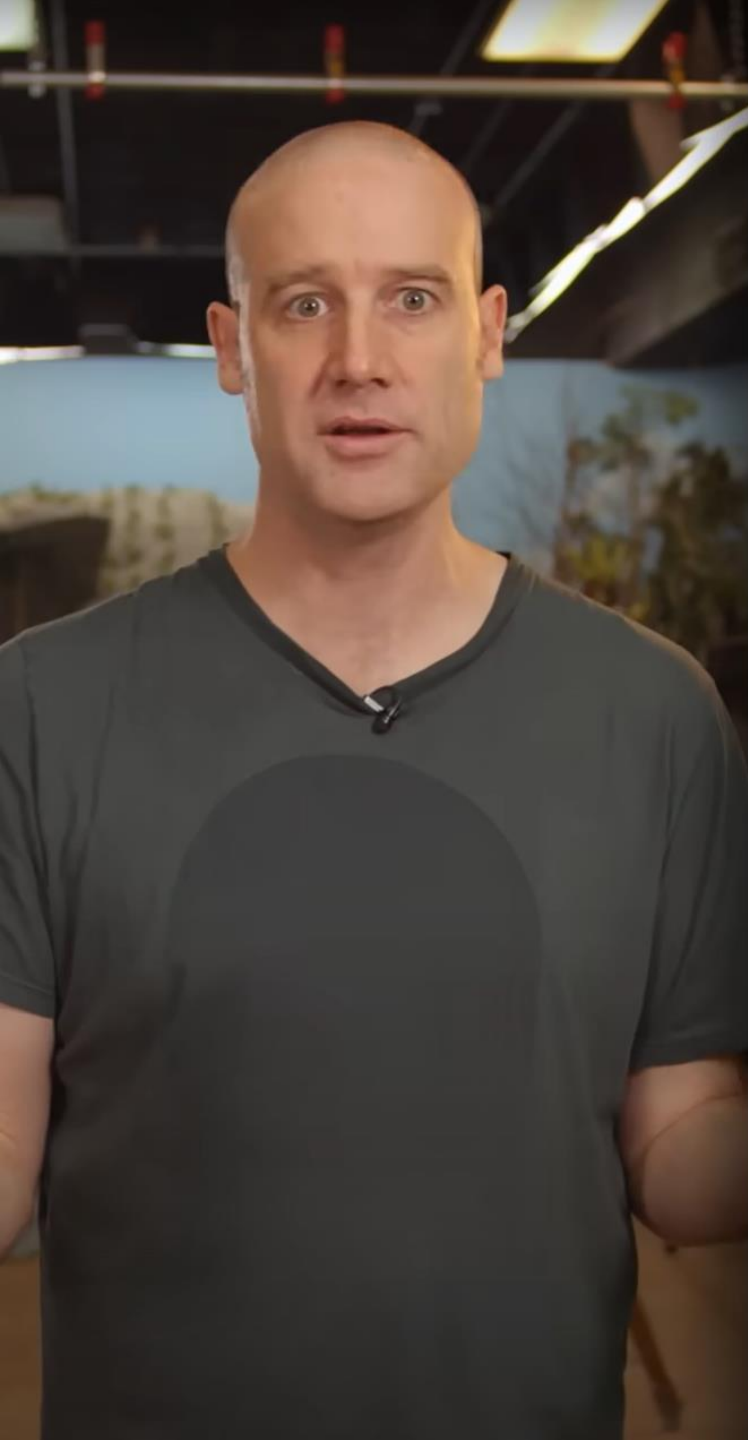
the biggest means having the largest volume

Vertices (points) of *the biggest* shape lie on the spherical surface of *the universe*

The *first shape discovered by computer* was thought to be
the biggest shape with **8** or less vertices

(this was not proven at the time the computer search paper was written in 1962)





29	11	11	49	15	15
30	11	10	50	15	14

Battersea College of Technology
London, S.W. 11, England

1. H. E. SALZER, "Equally weighted quadrature formulas over semi-infinite and infinite intervals," *J. Math. Phys.*, v. 34, 1955, p. 54-63.

Search For Largest Polyhedra

Paper
written in
1962

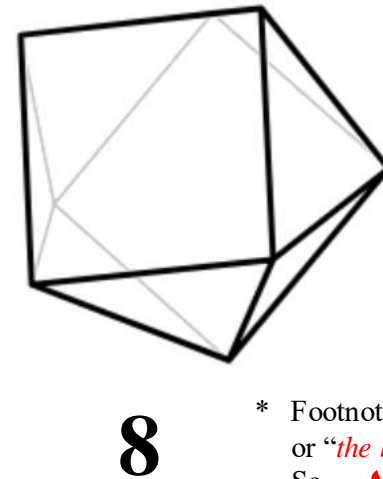
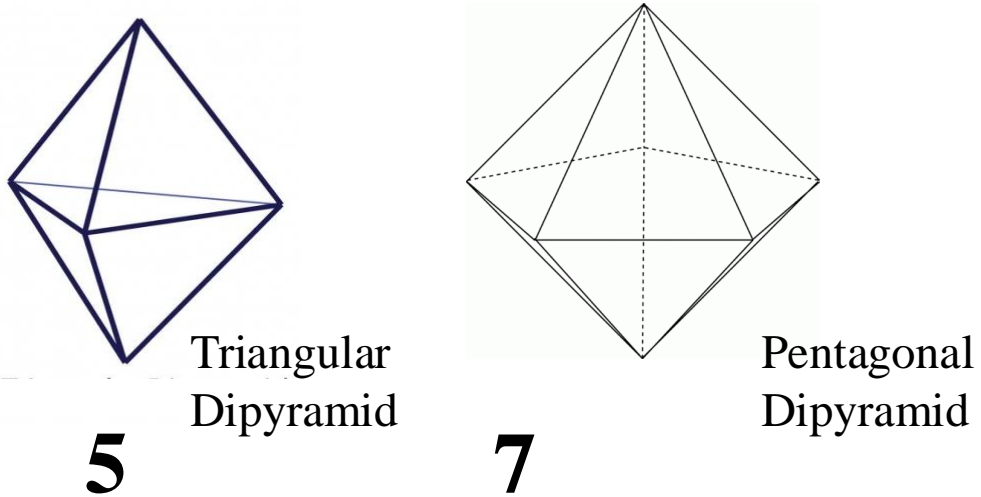
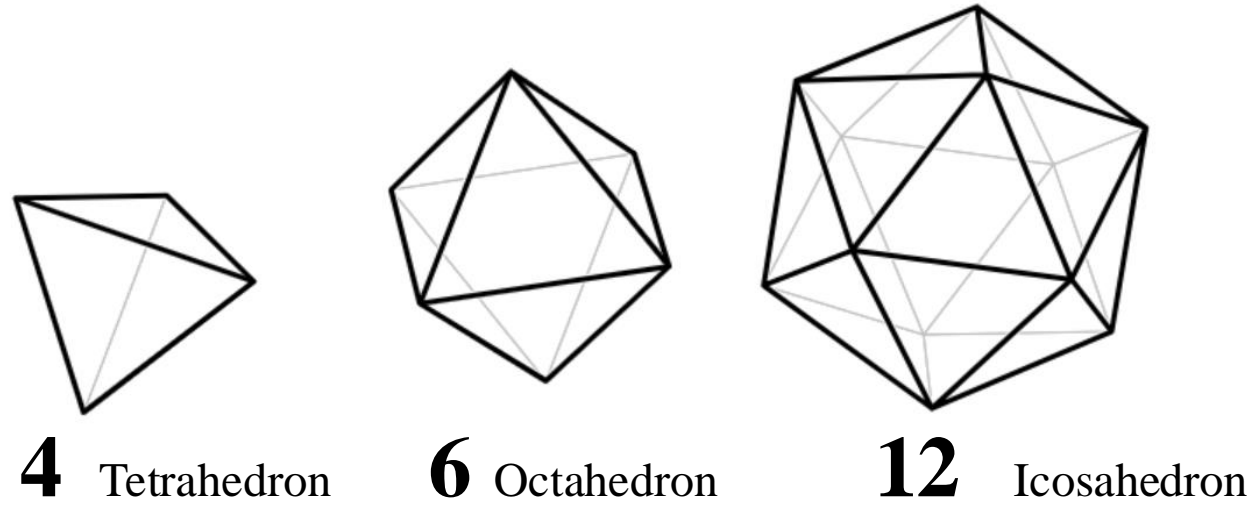
By Donald W. Grace

The configuration of eight points on the unit sphere which determines the convex polyhedron of maximum volume is not known. It is not the cube, whose volume is $8\sqrt{3}/9$, or about 1.5396, since the volume of the double pyramid (one point at each of the poles and the other six distributed uniformly around the equator) is $\sqrt{3}$, or about 1.732. [1, page 7]. The purpose of the present work was to determine if an even larger polyhedron could be found.

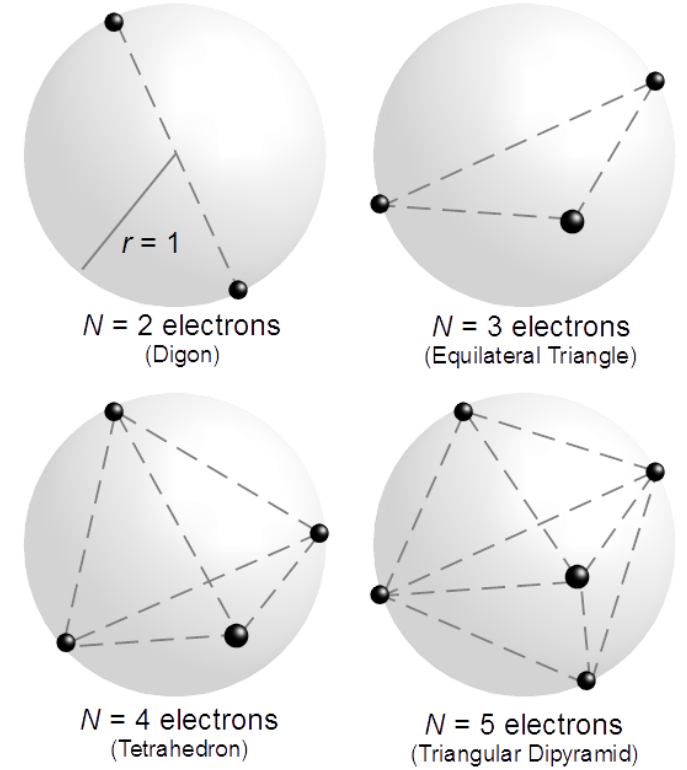
The search was carried out using gradient methods on the Burroughs 220 computer. Each set of eight points on the sphere may be interpreted as a vector, W , in Euclidean 16-space. For each W , let $V(W)$ denote the volume of the convex hull of the eight points, and $G(W)$ the gradient of $V(W)$. A starting configuration was introduced into the computer and modified iteratively. At each iteration, the 16-dimensional vector, W , was replaced by $W' = W + M \cdot G(W)$, for some real M

Constructing Polyhedra from Repelling Points on a Sphere

by [Simon Tatham](#), mathematician and programmer



Solutions of the Thomson Problem



* Footnote this isn't the first shape discovered by computer or "*the biggest shape in the universe*". And it isn't pink. So ... **✗** No repelling points is not how you get the maximum volume shape.

```

import libraries
import numpy as np # numpy to stack vectors in matrices
import random # random to generate random numbers
from math import pi, sin, cos, asin # maths functions used
import scipy.spatial as sp # sci py algorithms for calculating convex hulls , spatial transformation etc.
import scipy.optimize as op # sci py algorithms for optimising on several variables
import matplotlib.pyplot as pp # mathplot lib to plot shapes including polyhedra

# Function to calculate the volume of a polyhedron given its vertices
def polyhedron_volume(vertices):
    hull = sp.ConvexHull(vertices)
    return hull.volume

# Function to calculate a vertices matrix from a list of angles
def angle_vertices(angle):
    vert = np.empty((0,3))
    for i in range(int(len(angle)/2)):
        vert = np.vstack([[vert,[cos(angle[2*i])*cos(angle[2*i+1]), sin(angle[2*i])*cos(angle[2*i+1]), sin(angle[2*i+1])]]])
    return(vert)

# function to create n pairs of random angles from 0 - 2 pi
def angle_random(n):
    angle = []
    for i in range(n):
        theta = random.random() * 2 * pi
        phi = asin(random.random() * 2 - 1)
        angle.append(theta)
        angle.append(phi)
    return(angle)

# function to return the polyhedron volume given the angles
def volume(angle):
    return (polyhedron_volume(angle_vertices(angle)))

# function to plot the shape
def plot(vert,hull):
    pp.rcParams["figure.figsize"] = [10.00, 10.00]
    pp.rcParams["figure.autolayout"] = True
    fig = pp.figure()
    ax = fig.add_subplot(projection='3d')
    ax.plot(vert[:,0],vert[:,1],vert[:,2],"ko")
    for sx in hull.simplices:
        sx = np.append(sx,sx[0])
        pp.plot(vert[sx,0],vert[sx,1],vert[sx,2], 'k-')
    ax.set_axis_off()
    pp.show()

def compare(this_volume,local_maxima):
    if local_maxima == []:
        return True
    for i in range(len(local_maxima)):
        how_close = abs(local_maxima[i]-this_volume)
        if (how_close <= 0.0000001):
            return False
    return True

#run for n vertices and collect successive maximum volumes listing the local maxima found
n = 8
local_maxima = []

for i in range(1,100):
    angle = angle_random(n)
    bounds = []
    bounds = [(0, 2*pi),(-pi/2,pi/2)] * n
    result = op.minimize(lambda x: -volume(x), angle, bounds=bounds,tol=0.0000000000000000001)
    this_volume = volume(result.x)
    if compare(this_volume,local_maxima):
        local_maxima.append(this_volume)
    if this_volume == max(local_maxima):
        result_max = result

optimised_angles = result_max.x
optimised_vertices = angle_vertices(optimised_angles)

# output the angles , vertices and volume for the maxium volume polyhedron found
print(optimised_angles)
print()
print(optimised_vertices)
print()
print(volume(optimised_angles))
print()
print(local_maxima)

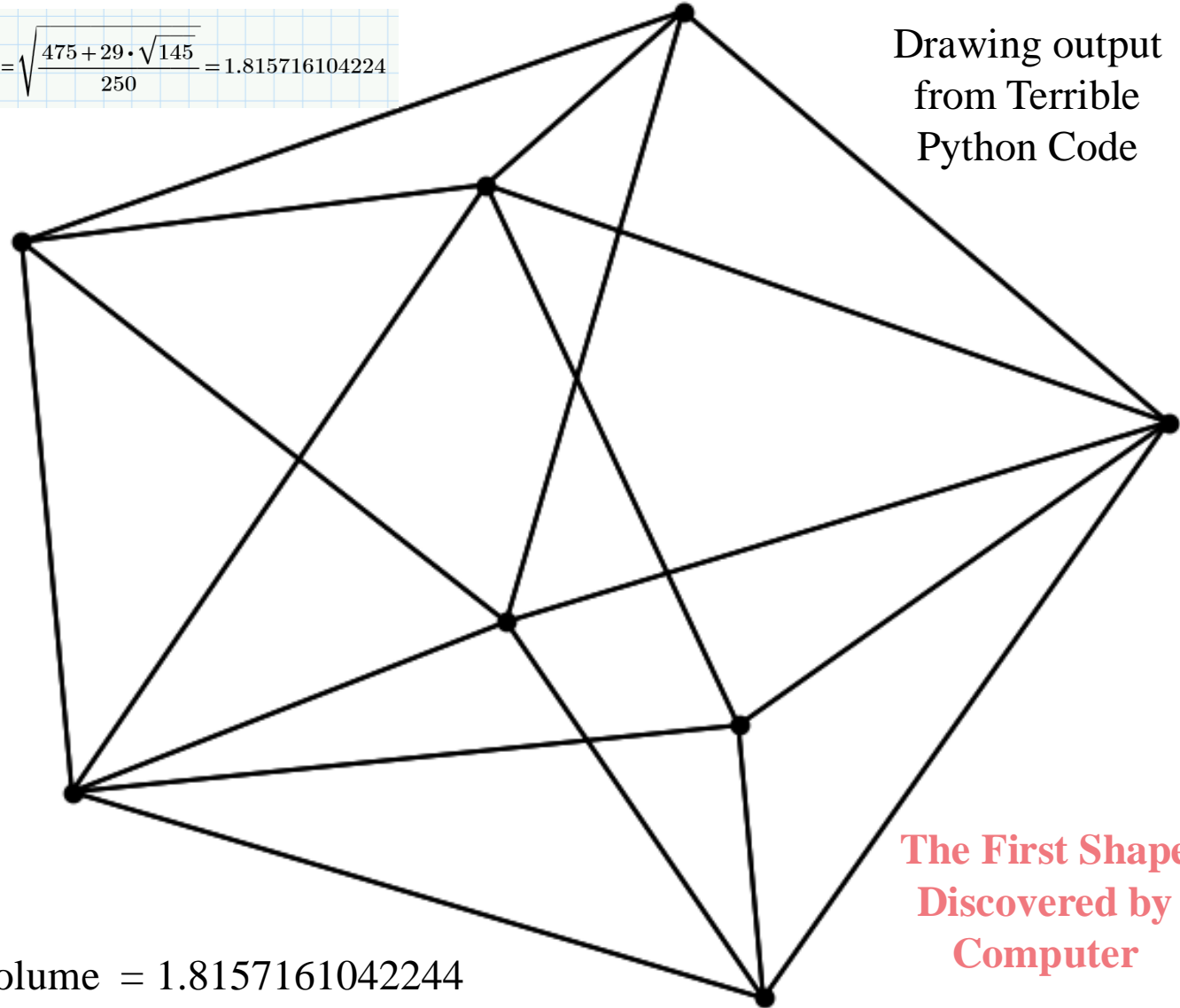
# Plot the shape
plot(optimised_vertices,sp.ConvexHull(optimised_vertices))

```

Really Terrible* Python Code

(* I tried to get the Microsoft Copilot AI to write this. It didn't work. It did however help me to figure out how to write the code above which does)

$$V_8 := \sqrt{\frac{475 + 29 \cdot \sqrt{145}}{250}} = 1.815716104224$$



Drawing output
from Terrible
Python Code

Volume = 1.8157161042244

Coordinates from 1970 paper giving a proof

- $p_1 = (\sin 3\varphi, 0, \cos 3\varphi),$
- $p_2 = (\sin \varphi, 0, \cos \varphi),$
- $p_3 = (-\sin \varphi, 0, \cos \varphi),$
- $p_4 = (-\sin 3\varphi, 0, \cos 3\varphi),$
- $p_5 = (0, -\sin 3\delta, -\cos 3\delta),$
- $p_6 = (0, -\sin \delta, -\cos \delta),$
- $p_7 = (0, \sin \delta, -\cos \delta),$
- $p_8 = (0, \sin 3\delta, -\cos 3\delta),$

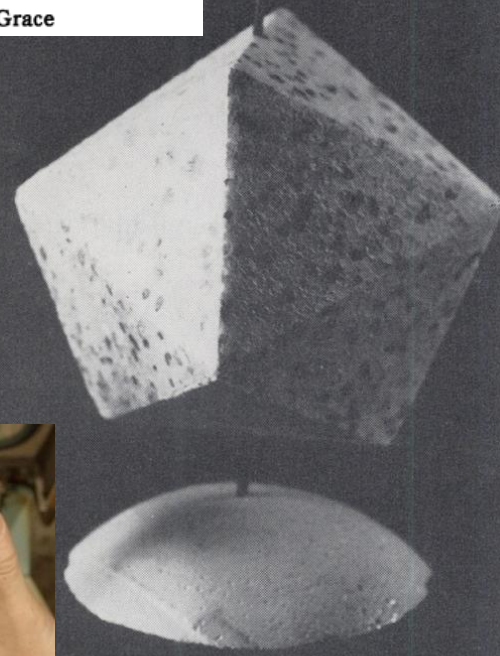
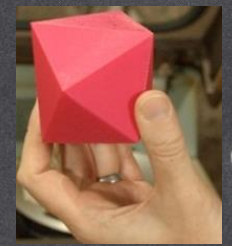
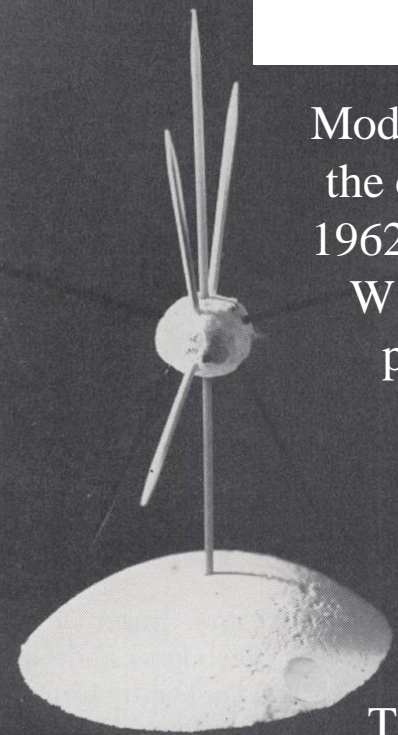
Volumes of Polyhedra Inscribed
in the Unit Sphere in E^3

JOEL D. BERMAN and KIT HANES

Math. Ann. 188, 78—84 (1970)
© by Springer-Verlag 1970

The First Shape
Discovered by
Computer

Search For Largest Polyhedra
By Donald W. Grace



Models from
the original
1962 Donald
W Grace
paper

The Pink 3D print

```
[ 0.71804608 -1.14191833 6.26256003 0.11615005 5.0577219 0.65812082
1.34537349 -0.01699593 3.70101673 0.32024262 1.93181758 1.11398967
2.70456585 -0.40059248 4.70860115 -0.6730715 ]
```

```
[[ 0.3131741 0.27359417 -0.90943291]
[ 0.9930509 -0.02048485 0.11588907]
[ 0.26780979 -0.74443613 0.61163123]
[ 0.22348624 0.97455891 -0.01699511]
[-0.80447011 -0.50371696 0.31479685]
[-0.15580419 0.41265063 0.8974656 ]
[-0.83428503 0.38973912 -0.38996399]
[-0.00296174 -0.781905 -0.62339056]]
```

1.8157161042244154

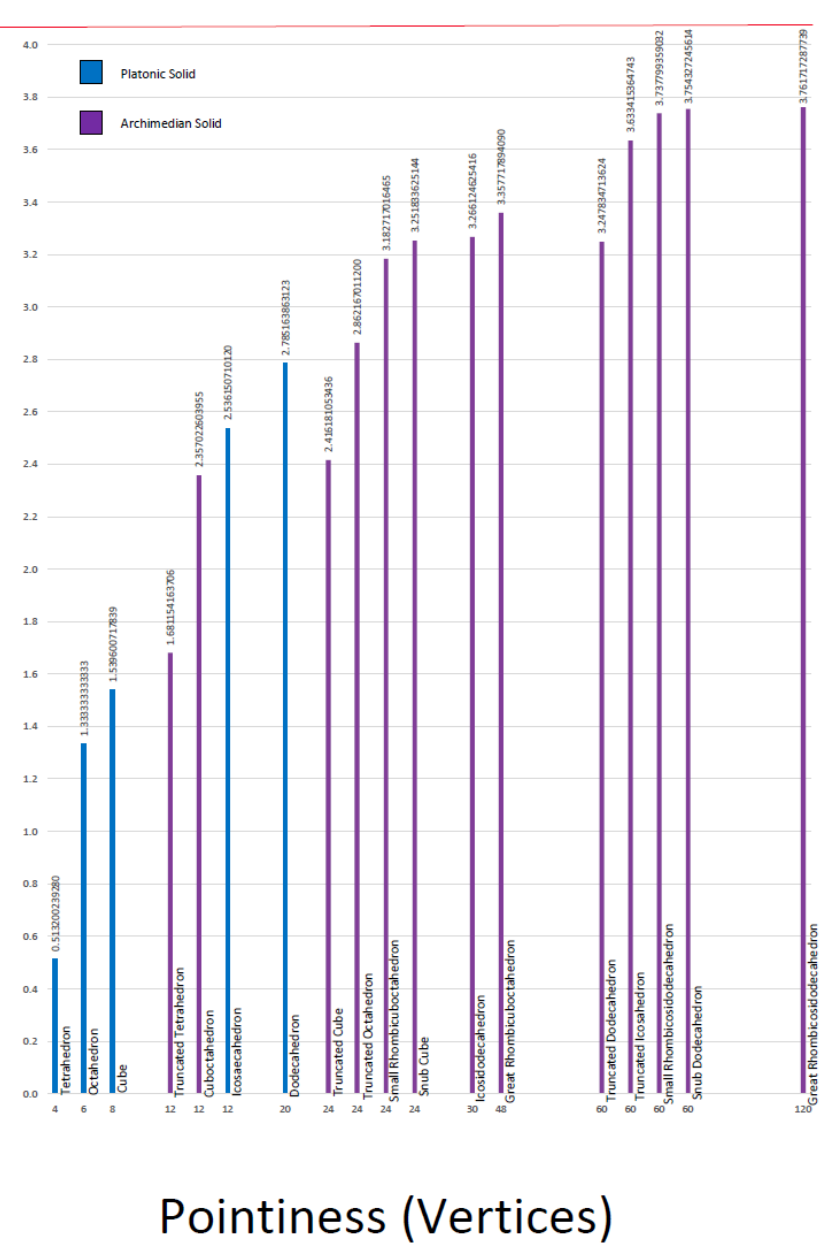
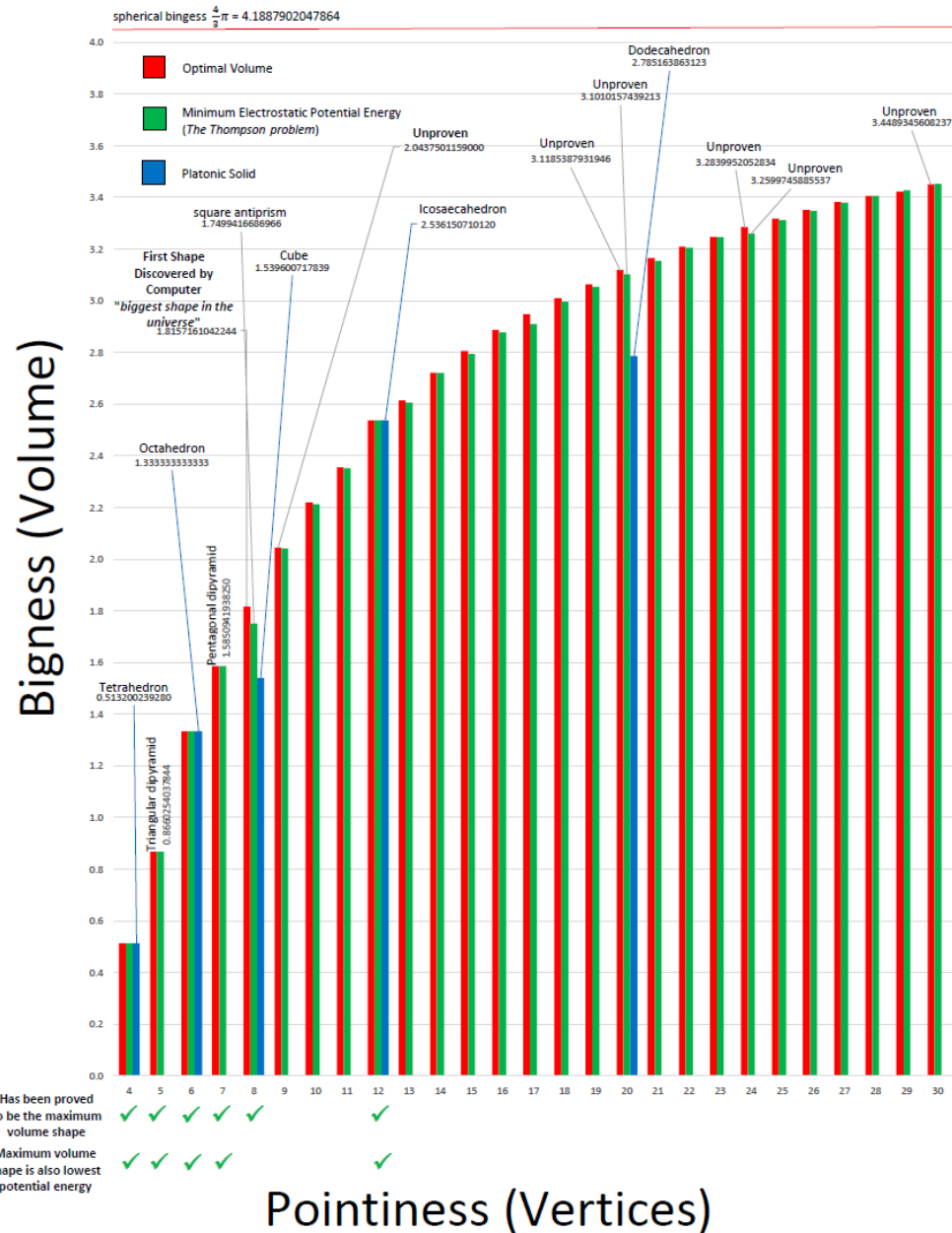
```
[1.7320508075688739, 1.8157161042244154, 1.77621489467476, 1.8033196842445272,
1.626289314909613, 1.7427840329120292]
```

Terrible Python
Code Output

Extra Slides for Discussion After the 2 ½ Minutes

More data from Python search and considering higher dimensional n-spheres

Bigness and Pointiness of Polyhedrons with Points on a Sphere

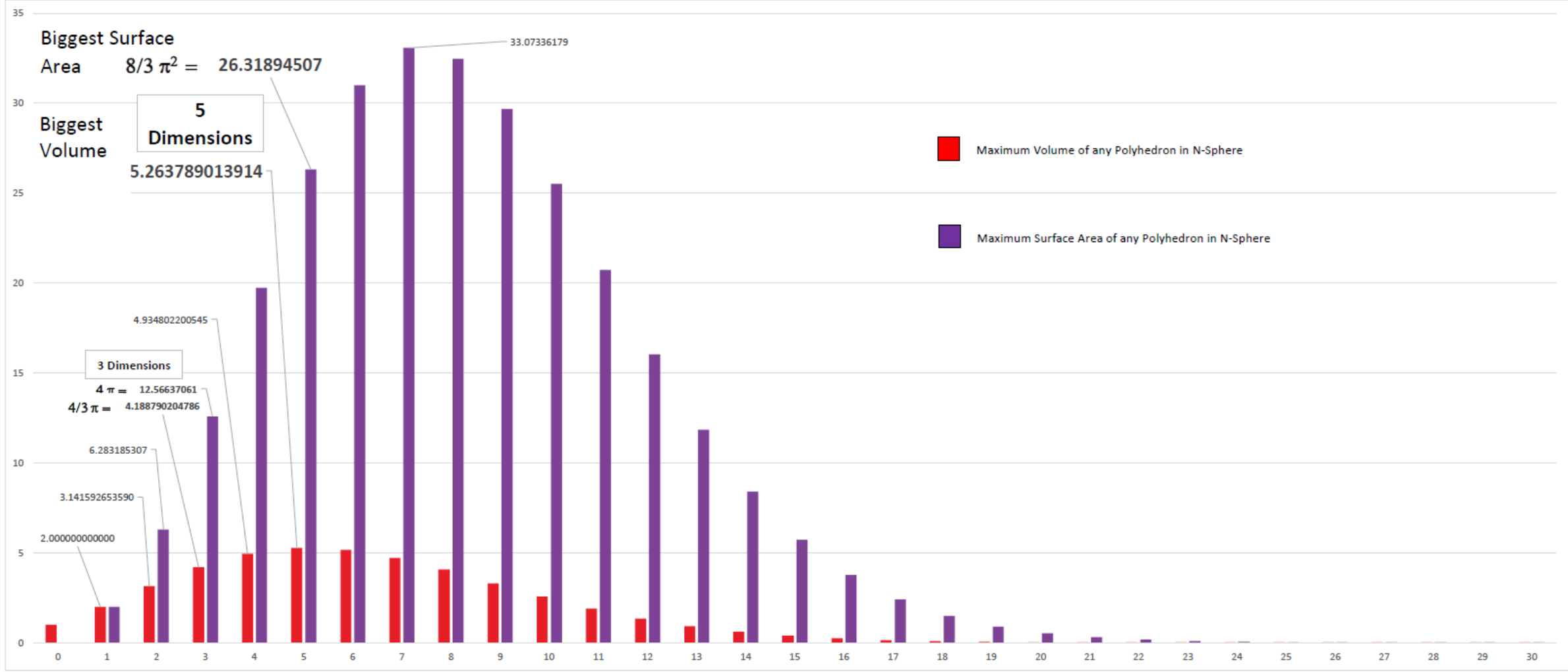


Python Data Comparison

Vertices	Faces	Edges	Neil Sloane 1994 Values	Difference To 1994 values	Hull Optimised Maximum Volume	% Vol of Sphere	Hull Repelled Thompson	% Vol of Sphere	Extra Volume	Exact Known Maximum Volume	Platonic Volume	% Vol of Sphere	Names
∞			4.1887902047864		4.1887902047864	100.00%	4.1887902047864	100.00%	0.00%	4.188790204786			sphere
2	1	1											diameter line
3	2	3											triangle
4	4	6	0.5132002392800	0.000000000003	0.5132002392797	12.25%	0.5132002392797	12.25%	0.00%	0.513200239280	0.513200239280	12.25%	tetrahedron
5	6	9	0.8660254037840	0.000000000004	0.8660254037844	20.67%	0.8660254037844	20.67%	0.00%	0.866025403784			triangular dipyramid
6	8	12	1.3333333333330	0.000000000003	1.3333333333333	31.83%	1.3333333333333	31.83%	0.00%	1.333333333333	1.333333333333	31.83%	octahedron
7	10	15	1.5850941938250	0.000000000000	1.5850941938250	37.84%	1.5850941938253	37.84%	0.00%	1.585094193825			pentagonal dipyramid
8	12	18	1.8157161042240	0.000000000004	1.8157161042244	43.35%	1.7499416686966	41.78%	3.76%	1.815716104224	1.539600717839	36.76%	square antiprism / cube
9	14	21	2.0437501159000	0.000000000000	2.0437501159000	48.79%	2.0396301518616	48.69%	0.20%	2.043750115900			
10	16	24	2.2187111315450	0.000000000000	2.2187111315450	52.97%	2.2126310997272	52.82%	0.27%	$V_9 := 3 \cdot \sqrt{2 \cdot \sqrt{3}} - 3 = 2.04375011589964$			Exact Volume Formula not proven to be maximum
11	18	27	2.3546344950690	0.000000000000	2.3546344950690	56.21%	2.3510386032607	56.13%	0.15%				
12	20	30	2.5361507101200	0.000000000004	2.5361507101204	60.55%	2.5361507101204	60.55%	0.00%	2.536150710120	2.536150710120	60.55%	icosahedron
13	22	33	2.6128341520600	0.000000000002	2.6128341520602	62.38%	2.6052614240915	62.20%	0.29%				
14	24	36	2.7209778993490	0.000000000000	2.7209778993490	64.96%	2.7199499431799	64.93%	0.04%				
15	26	39	2.8043793818350	0.000000000000	2.8043793818350	66.95%	2.7933206638847	66.69%	0.40%				
16	28	42	2.8864553922750	0.000000000005	2.8864553922745	68.91%	2.8773301010496	68.69%	0.32%				
17	30	45	2.9475229848080	0.000000000000	2.9475229848080	70.37%	2.9091108480200	69.45%	1.32%				
18	32	48	3.0096132525230	0.000000000006	3.0096132525225	71.85%	2.9948901860718	71.50%	0.49%				
19	34	51	3.0632162779160	0.000000000005	3.0632162779155	73.13%	3.0538431674382	72.91%	0.31%				
20	36	54	3.1185387931950	0.000000000004	3.1185387931946	74.45%	3.1010157439213	74.03%	0.57%		2.785163863123	66.49%	dodecahedron
21	38	57	3.1644416167890	0.000000000004	3.1644416167886	75.55%	3.1540628953941	75.30%	0.33%				
22	40	60	3.2082399958820	0.000000000003	3.2082399958817	76.59%	3.2035885287718	76.48%	0.15%				
23	42	63	3.2469420244370	0.000000000008	3.2469420244362	77.52%	3.2448919775552	77.47%	0.06%				
24	44	66	3.2839952052830	0.000000000004	3.2839952052834	78.40%	3.2599745885537	77.83%	0.74%				
25	46	69	3.3162635225950	0.000000000003	3.3162635225947	79.17%	3.3101184851061	79.02%	0.19%				
26	48	72	3.3493598724940	0.000000000007	3.3493598724933	79.96%	3.3467742590855	79.90%	0.08%				
27	50	75	3.3804160382750	0.0000581105148	3.3803579277602	80.70%	3.3794495999647	80.68%	0.03%				
28	52	78	3.4073797227340	0.0017975356790	3.4055821870550	81.30%	3.4056021668058	81.30%	0.00%				
29	54	81	3.4309531102130	0.0095162252381	3.4214368849749	81.68%	3.4260151053408	81.79%	-0.13%				
30	56	84	3.4551257520620	0.0061911912383	3.4489345608237	82.34%	3.4506408767053	82.38%	-0.05%				

Bigness and Pointiness limit for polyhedrons in n-spheres

Bigness (Volume, Surface Area)



Pointiness (Space Dimensions in Universe)

Volume and Surface Area Calculations for n-spheres

N	1 + N/2	$\Gamma(1 + N/2)$	V(N)	N/2	$\Gamma(N/2)$	S(N-1)
0	1		1 1.000000000000 =	0		0 = 0
1	1.5	0.886226925453	2.000000000000 =	0.5	1.772453850906	2 = 2
2	2		1 3.141592653590 =	π		1 6.28318531 = 2
3	2.5	1.329340388179	4.188790204786 =	π	0.886226925453	12.5663706 = 4
4	3		2 4.934802200545 =	π^2		1 19.7392088 = 2
5	3.5	3.323350970448	5.263789013914 =	π^2	1.329340388179	26.3189451 = 8 / 3
6	4		6 5.167712780050 =	π^3		2 31.0062767 =
7	4.5	11.631728396567	4.724765970331 =	π^3	3.323350970448	33.0733618 = 16 / 15
8	5		24 4.058712126417 =	π^4		6 32.469697 = 1 / 3
9	5.5	52.342777784554	3.298508902739 =	π^4	11.631728396567	29.6865801 = 32 / 105
10	6		120 2.550164039877 =	π^5		24 25.5016404 = 1 / 12
11	6.5	287.885277815044	1.884103879390 =	π^5	52.342777784554	20.7251427 = 64 / 945
12	7		720 1.335262768855 =	π^6		120 16.0231532 = 1 / 60
13	7.5	1871.254305797790	0.910628754783 =	π^6	287.885277815044	11.8381738 = 128 / 10395
14	8		5040 0.599264529321 =	π^7		720 8.38970341 = 1 / 360
15	8.5	14034.407293483400	0.381443280823 =	π^7	1871.254305797790	5.72164921 = 256 / 135135
16	9		40320 0.235330630359 =	π^8		5040 3.76529009 = 1 / 2520
17	9.5	119292.461994609000	0.140981106917 =	π^8	14034.407293483400	2.39667882 = 512 / 2027025
18	10		362880 0.082145886611 =	π^9		40320 1.47862596 = 1 / 20160
19	10.5	1133278.388948790000	0.046621601030 =	π^9	119292.461994609000	0.88581042 = 1024 / 34459425
20	11		3628800 0.025806891390 =	π^{10}		362880 0.51613783 = 1 / 181440
21	11.5	11899423.083962200000	0.013949150409 =	π^{10}	1133278.388948790000	0.29293216 = 2048 / 654729075
22	12		39916800 0.007370430946 =	π^{11}		3628800 0.16214948 = 1 / 1814400
23	12.5	136843365.465566000000	0.003810656387 =	π^{11}	11899423.083962200000	0.0876451 = 4096 / 13749310575
24	13		479001600 0.001929574309 =	π^{12}		39916800 0.04630978 = 1 / 19958400
25	13.5	1710542068.319570000000	0.000957722409 =	π^{12}	136843365.465566000000	0.02394306 = 8192 / 316234143225
26	14		6227020800 0.000466302806 =	π^{13}		479001600 0.01212387 = 1 / 239500800
27	14.5	23092317922.314200000000	0.000222872125 =	π^{13}	1710542068.319570000000	0.00601755 = 16384 / 7905853580625
28	15		87178291200 0.000104638105 =	π^{14}		6227020800 0.00292987 = 1 / 3113510400
29	15.5	334838609873.556000000000	0.000048287823 =	π^{14}	23092317922.314200000000	0.00140035 = 32768 / 213458046676875
30	16		1307674368000 0.000021915353 =	π^{15}		87178291200 0.00065746 = 1 / 43589145600

Volume and Surface Area For Regular Polytopes with points on the surface of an N-Sphere in 1 - 30 Dimensions

N	n-simplex side	n-simplex volume	n-orthoplex side	n-orthoplex volume	n-cube side	n-cube volume	n-polytope side	n-polytope volume	n-polytope side	n-polytope volume	n-polytope side	n-polytope volume
0	0.000000000000	0.0000000000000000	0.000000000000	0.0000000000000000	0.000000000000	0.0000000000000000						
1	2.000000000000	2.0000000000000000	1.414213562373	2.0000000000000000	2.000000000000	2.0000000000000000						
2	1.732050807569	1.2990381056766600	1.414213562373	2.0000000000000000	1.414213562373	2.0000000000000000						
3	1.632993161855	0.5132002392796670	1.414213562373	1.3333333333333300	1.154700538379	1.5396007178390000	Icosahedron		Dodecahedron			
4	1.581138830084	0.1455773422851430	1.414213562373	0.6666666666666600	1.000000000000	1.0000000000000000	1.051462224238	2.536150710120	0.713644179546	2.785163863123		
5	1.549193338483	0.0321993788759970	1.414213562373	0.2666666666666670	0.894427191000	0.5724334022399460	1.000000000000	2.000000000000	0.618033988750	3.862712429687	0.270090756738	4.192627457812
6	1.527525231652	0.0058352154044184	1.414213562373	0.0888888888888888	0.816496580928	0.2962962962962970	Icositetrachoron 24-cell		Hexacosichoron 600-cell		Hecatonicosachoron 120-cell	
7	1.511857892037	0.0008955426451049	1.414213562373	0.0253968253968254	0.755928946018	0.1410479666040260						
8	1.500000000000	0.0001191820417132	1.414213562373	0.0063492063492063	0.707106781187	0.0624999999999999						
9	1.490711985000	0.0000140005686247	1.414213562373	0.0014109347442681	0.6666666666667	0.0260122948737489						
10	1.483239697419	0.0000014719624645	1.414213562373	0.0002821869488536	0.632455532034	0.0102400000000000						
11	1.477097891752	0.0000001400464130	1.414213562373	0.0000513067179734	0.603022689156	0.0038341595748878						
12	1.471960144388	0.0000000121676680	1.414213562373	0.0000085511196622	0.577350269190	0.0013717421124829						
13	1.467598771411	0.0000000009727117	1.414213562373	0.0000013155568711	0.554700196225	0.0004707151254045						
14	1.463850109423	0.0000000000720078	1.414213562373	0.0000001879366959	0.534522483825	0.0001554260068995						
15	1.460593486680	0.0000000000049634	1.414213562373	0.0000000250582261	0.516397779494	0.0000495182732539						
16	1.457737973711	0.0000000000003201	1.414213562373	0.0000000031322783	0.500000000000	0.0000152587890625						
17	1.455213750218	0.0000000000000194	1.414213562373	0.0000000003685033	0.485071250073	0.0000045571580884						
18	1.452966314514	0.0000000000000011	1.414213562373	0.0000000000409448	0.471404520791	0.0000013215614934						
19	1.450952500220	0.0000000000000001	1.414213562373	0.0000000000043100	0.458831467741	0.0000003727440405						
20	1.449137674619	0.0000000000000000	1.414213562373	0.0000000000004310	0.447213595500	0.0000001024000000						
21	1.447493728911	0.0000000000000000	1.414213562373	0.0000000000000410	0.436435780472	0.0000000274364119						
22	1.445997610962	0.0000000000000000	1.414213562373	0.0000000000000037	0.426401432711	0.0000000071781151						
23	1.444630237029	0.0000000000000000	1.414213562373	0.0000000000000003	0.417028828114	0.0000000018357764						
24	1.443375672974	0.0000000000000000	1.414213562373	0.0000000000000000	0.408248290464	0.0000000004593937						
25	1.442220510186	0.0000000000000000	1.414213562373	0.0000000000000000	0.400000000000	0.0000000001125900						
26	1.441153384246	0.0000000000000000	1.414213562373	0.0000000000000000	0.392232270276	0.0000000000270475						
27	1.440164599646	0.0000000000000000	1.414213562373	0.0000000000000000	0.384900179460	0.0000000000063738						
28	1.439245834258	0.0000000000000000	1.414213562373	0.0000000000000000	0.377964473009	0.0000000000014744						
29	1.438389904456	0.0000000000000000	1.414213562373	0.0000000000000000	0.371390676354	0.0000000000003350						
30	1.437590576857	0.0000000000000000	1.414213562373	0.0000000000000000	0.365148371670	0.0000000000000748						